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THE SYMBOLIC DYNAMICS OF THE TENT MAP

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1. Introduction

In 1987, E. W. Hobson [2] called the triangle T' formed from the feet of the altitudes of a triangle T the *pedal triangle*. Recently J. G. Kingston and J. L. Synge [3] revisited and corrected Hobson's work and, for example, determined a criterion for some pedal iterate of T to have the same angles as T .

In 1993, J. C. Alexander [1] revisited the issue again from a different mathematical point of view, which makes it routine to understand the behavior of the angles of successive pedal triangles. For each triangle T, we assign a label $E(T)$ that makes the behavior under the iterated pedal map obvious. Equally important, there is a straightforward way of determining the angles of the triangle from its label. More precisely, consider all infinite sequence $a_1 a_2 a_3 \cdots$ of four symbols, say each a_i equals 0,1,2 or 3. Every triangle T is labelled with one such sequence $E(T)$, and two triangles are labelled with the same sequence if and only if they are similar. Moreover, if $E(T) = a_1 a_2 a_3 \cdots$, then $E(T') = a_2 a_3 a_4 \cdots$ obtained by erasing the first symbol in $E(T)$. Then if for example, $013013013 \cdots$ (infinitely repeating) is the label of a triangle T, it is clear that the third pedal iterate of T is similar to T.

J. C. Alexander [1] showed that almost all labels correspond to a triangle, and he also gave an algorithm to find the triangle from its label.

This type of labelling is the subject of *symbolic dynamics*, which is quite powerful when it works.

In this paper, we study the symbolic dynamics of the tent map (see 2-2 for detailed definitions). And we show that all labels correspond to a point in $I = [0,1]$, and we give a coding algorithm to find the point in I from its label.

We have the following main results.

THEOREM 3.1.1. *There exists an encoding for the tent map.*

THEOREM 3.2.1. *There exists a decoding for the tent map.*

THEOREM 3.3.1. *The number of periodic points of prime period n under the tent map is $2^n - 2$.*

THEOREM 3.3.4. *There exists a point $x \in I$ such that the orbit $\{f^n(x)\}$ is dense in I.*

2. Preliminaries

Let $f : X \to X$ be a continuous function from a space X into itself. For each positive integer n, f^n denotes n-th iterate of f, that is, f^0 is the identity and $f^{n+1} = f \circ f^n$ for each $n \geq 1$. A point $x \in X$ is called a *periodic point* of f if $f^n(x) = x$ for some positive integer n. In this case, the least such n is called the *period* of x. A point $x \in X$ is called an ***eventually periodic point*** of f if there exists a positive integer n such that $f^n(x)$ is periodic. A point $x \in X$ is called a *fixed point* if $f(x) = x$, that is, x is a point of period one. The orbit of x is the set $\{f^k(x) | k = 1, 2, \cdots\}$, and denoted by $\mathrm{Orb}(x)$.

2-1. Symbolic Dynamics

Consider the n symbols, namely $0, 1, 2, 3, \cdots, n-1$.

Out of these we form infinite *words* $a = a_1 a_2 a_3 \cdots$, where each a_i is one of the symbols $0, 1, 2, 3, \cdots, n-1$. The set of all such words is called the *sequence space* S_n on the n symbols.

There is a transformation G on S_n , namely the *shift* on n symbols :

$$G : a_1 a_2 a_3 \cdots \mapsto a_2 a_3 a_4 \cdots .$$

We use the standard notation $\overline{a_1 a_2 a_3 \cdots a_r}$ to mean the infinitely repeating word $a_1 a_2 a_3 \cdots a_r a_1 a_2 a_3 \cdots a_r a_1 a_2 a_3 \cdots a_r \cdots$.

More generally, $a_1 a_2 a_3 \cdots a_k \overline{a_{k+1} a_{k+2} a_{k+3} \cdots a_{k+r}}$ denote the word

$$a_1 a_2 a_3 \cdots a_k a_{k+1} a_{k+2} a_{k+3} \cdots a_{k+r} a_{k+1} a_{k+2} a_{k+3} \cdots a_{k+r} \cdots .$$

that is eventually periodic under the shift.

Let $f : X \to X$ be a function from a set X into X. To represent f by the shift G on n symbols means to assign to each $x \in X$ a word $E(x) \in S_n$ so that $G \circ E = E \circ f$. Then such an assignment E is called the *encoding* for f. The encoding of $f(x)$ is the shift of the encoding of x.

The opposite process $D : S_n \to X$ of a assigning an element of X to each word is called the *decoding* for f. To clarify all this abstraction, we present an example. Consider the set X of reals x, $0 \le x < 1$ with the transformation $f(x) = 2x (\text{mod } 1)$.

We find an encoding E . Write any $x \in X$ in binary notation

$$x = .x_1x_2x_3\cdots, \quad \text{each } x_i = 0 \text{ or } 1.$$

Then $f(x)$ has the binary representation $f(x) = .x_2x_3x_4\cdots$ (multiply by 2 (= 10 in binary) and drop the integral part).

Thus the encoding E that takes x to its string of binary digits represents f as the shift on 0 and 1. It is simple to decode, that is , go from a word $x_1x_2x_3\cdots$ to the represented number :

$$D(x_1x_2x_3\cdots) = x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

Note that the two sequences $.x_1x_2\cdots 0_k\overline{1}$ and $.x_1x_2\cdots 1_k\overline{0}$ are binary representations of the same number. We choose one of them satisfying our purpose.

2-2. The tent map

For $I = [0,1]$, the *tent map* $f : I \to I$ is a continuous function defined by

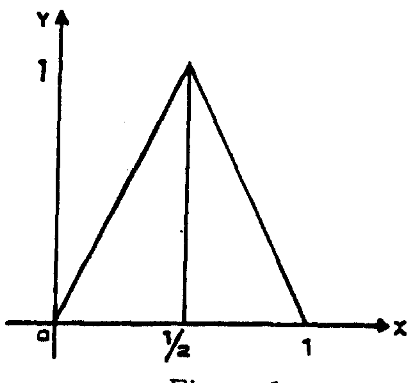

Figure 1.

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le \dfrac{1}{2} \\ 2(1-x) & \text{if} \quad \dfrac{1}{2} < x \le 1. \end{cases}$$

Consider the tent map $f: I \to I$. Write any $x \in I$ in a binary notation

$$x = .x_1x_2x_3\cdots \quad \text{where } x_i = 0 \text{ or } 1.$$

If $x_1 = 0$, then $0 \le x \le \frac{1}{2}$. Hence $f(x)$ has a binary representation $f(x) = .x_2x_3x_4\cdots$ as the example in the previous section shows.

Suppose $x_1 = 1$. In this case, if $x_2 = x_3 = x_4 = \cdots = 0$ then $x = \frac{1}{2}$. We can write $\frac{1}{2} = .0\overline{1}$ and we know that $f(\frac{1}{2}) = 1 = .\overline{1}$. If some of x_n is not zero for $n \ge 2$, then $\frac{1}{2} < x \le 1$. In this case, $f(x)$ has a binary representation $.(1-x_2)(1-x_3)(1-x_4)\cdots$.

By combining the above results, we conclude that $f(x)$ has a binary representation

$$f(x) = \begin{cases} .x_2x_3\cdots & \text{if} \quad x_1 = 0 \\ .x_2'x_3'\cdots & \text{if} \quad x_1 = 1, \end{cases}$$

where $x_i' = (1-x_i)$, the complement of x_i.

3. Coding Algorithms for the Tent Map

In this section, we investigate an encoding algorithm of a point x and decoding algorithm of a word a for the tent map.

3-1. An encoding algorithm

Let $f: I \to I$ be the tent map. We construct an encoding with the binary representation of a point $x \in I$ for the tent map f. Now we define subinterval I_i of I by $I_1 = [0, \frac{1}{2}]$ and $I_2 = (\frac{1}{2}, 1]$.

For each $x \in I$, the label of x with respect to f is the infinite sequence $a_1a_2a_3\cdots$, where a_i is the interval label of the n-th iterate of f, that is, the word of x with respect to f is the infinite sequence $a_1a_2a_3\cdots$ defined by $a_n = i$ if and only if $f^{n-1}(x) \in I_i$ ($i = 1$ or 2) for each $n = 1, 2, 3, \cdots$.

It is convenient to string them together to form an infinite word

$$a = a_1a_2a_3\cdots.$$

For $x \in I$, the word of $f(x)$ is the shift sequence $a_2a_3a_4\cdots$ obtained by erasing the first symbol of the word a of x.

To algorithmically encode ponits of I , at first, we write any point $x \in I$ in binary notation $x = .x_1x_2\cdots$, where $x_1x_2\cdots$ is a string of $0'$s and $1'$s.

The first digit a_1 of the sequence for x is determined as follow:

If $x_1 = 0$ then $x \in I_1$, $a_1 = 1$. Let $x_1 = 1$. If $x_2 = x_3 = \cdots = 0$, then $x = \frac{1}{2} \in I_1$ and $a_1 = 1$. Therefore if we set $\frac{1}{2} = .0\overline{1}$, then the encoding algorithm for $x = \frac{1}{2}$ is the same as the case $x_1 = 0$. If one of x_n's $(n \geq 2)$ is not zero, then $x \in I_2$ and $a_1 = 2$. Therefore we conclude that by putting $.0\overline{1}$ instead of $.1\overline{0}$ for the binary representation of $\frac{1}{2}$, the first digit a_1 of the sequence for $x = .x_1x_2\cdots$ is determined by

$$(*) \qquad a_1 = \begin{cases} 1 & \text{if} \quad x_1 = 0 \\ 2 & \text{if} \quad x_1 = 1. \end{cases}$$

To determine the second digit a_2 in the encoding of $x = .x_1x_2x_3\cdots$, we consider the following binary representation of $f(x)$:

$$f(x) = \begin{cases} .x_2x_3\cdots & \text{if} \quad x_1 = 0 \\ .x_2'x_3'\cdots & \text{if} \quad x_1 = 1, \end{cases}$$

where $x_i' = (1 - x_i)$, the complement of x_i.

Now, as the above method $(*)$ we give the second digit a_2 as follows :

$$a_2 = \begin{cases} 1 & \text{if} \quad x_2 \ (or\ x_2') = 0 \\ 2 & \text{if} \quad x_2 \ (or\ x_2') = 1. \end{cases}$$

Continuing this process, we can obtain the word $a = a_1a_2a_3\cdots$ for given any point $x \in I$.

As an immediate consequence of the above encoding algorithms, we have the following theorem.

THEOREM 3.1.1. *There exists an encoding* E *for the tent map* f, *that is the following diagram*

$$\begin{array}{ccc} I & \xrightarrow{\ f\ } & I \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ S_2 & \xrightarrow[G]{} & S_2 \end{array}$$

is commutative.

EXAMPLE 1. Starting with a point $\frac{1}{8}$ in binary notations $0.001\overline{0}$. However, we will use the same number $0.000\overline{1}$ instead of $0.001\overline{0}$.

Since $x_1 = 0$, we have $a_1 = 1$. Also since $x_1 = 0$, we know that $.x_2x_3x_4 \cdots = .00\overline{1}$. Hence $a_2 = 1$. By the same process $a_3 = 1$ and $a_4 = 2$. Now since $x_4 = 1$, we rewrite the sequence $.x_1x_2x_3x_4x_5 \cdots$ by $.0001\overline{0}$. And we know that $x_i = 0$ for all $i > 4$, hence we have the digits$a_5 = a_6 = \cdots = 1$. Thus the word of $\frac{1}{8}$ is $a = 1112\overline{1}$. Note that the n-th iterates of this point $\frac{1}{8}$ are

$$x = \frac{1}{8} \in I_1,\ f(x) = \frac{2}{8} \in I_1,\ f^2(x) = \frac{4}{8} \in I_1,\ f^3(x) = 1 \in I_2,\ f^4(x) = 0 \in I_1, \cdots$$

EXAMPLE 2. $a = 1\overline{111222222}$ is the word of $\frac{15}{365}$.

3-2. A decoding algorithm

To algorithmically decode a word, we go backwards through the encoding process in the previous section 3-1. It is straightforward to see that this leads to the following *decoding algorithm.*

A: Given a word $a = a_1a_2a_3\cdots$, for each digit a_i, associate a digit $x_i^{(1)}$ as follows:

$$x_i^{(1)} = \begin{cases} 0 & \text{if} \quad a_i = 1 \\ 1 & \text{if} \quad a_i = 2. \end{cases}$$

Concatenate these together to form an infinite sequence

$$P_1 = 0.x_1^{(1)}x_2^{(1)}x_3^{(1)}\cdots.$$

B: Inductively obtain a sequence

$$P_i = 0.x_1^{(i)}x_2^{(i)}x_3^{(i)}\cdots.$$

To obtain P_{i+1} from P_i, we consider the i-th element $x_i^{(i)}$ of P_i.

If $x_i^{(i)} = 0$, then $x_j^{(i+1)} = x_j^{(i)}$ for each $j = 1, 2, 3\cdots$,that is, $P_{i+1} = P_i$. If $x_i^{(i)} = 1$, then we put

$$x_j^{(i+1)} = \begin{cases} x_j^{(i)} & \text{if} \quad j \neq i+1 \\ 1 - x_j^{(i)} & \text{if} \quad j = i+1. \end{cases}$$

C: Note that $x_j^{(i+1)} = x_j^{(i)}$, for $j \leq i$. Thus there is a limit, which we denote

$$P = 0.x_1 x_2 x_3 \cdots$$

D: The point is determined by evaluating the binary expansions of the sequence.

As an immediate consequence of the above decoding algorithm, we obtain the following theorem

THEOREM 3.2.1. *There exists a decoding* $D : S_2 \to I$ *for the tent map* f, *that is, the following diagram*

$$\begin{array}{ccc} I & \xrightarrow{\;f\;} & I \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ S_2 & \xrightarrow[G]{} & S_2 \end{array}$$

is commutative.

EXAMPLE 3. Starting with the word $a = 212\overline{21}$.

According to our decoding algorithm we obtain the sequence $P_1 = 101\overline{10}$.

$$P_2 = 111\overline{10}, \quad P_3 = 1\overline{10}, \quad P_4 = 1\overline{10}, \quad P_5 = 11011\overline{10},$$
$$P_6 = 11011\overline{001}, \quad P_7 = 11011\overline{001}, \quad P_8 = 11011\overline{001}, \quad P_9 = 11011001\overline{110},$$
$$P_{10} = 110110011\overline{001}, \quad P_{11} = 110110011\overline{001}, \quad P_{12} = 110110011\overline{001}, \cdots$$

After proceeding this sequence we end up with the limit sequence

$$P = 110\overline{1100}.$$

Evaluating these binary expantions (by summing geometric series), we find that this word corresponds to the point $\frac{102}{120}$.

Note that the n-th iterates of the point $\frac{102}{120}$ is

$$x = \frac{102}{120} \in I_2 \quad f(x) = \frac{36}{120} \in I_1 \quad f^2(x) = \frac{72}{120} \in I_2 \quad f^3(x) = \frac{96}{120} \in I_2$$
$$f^4(x) = \frac{48}{120} \in I_1 \quad f^5(x) = \frac{96}{120} \in I_2 \quad f^6(x) = \frac{48}{120} \in I_1, \cdots$$

EXAMPLE 4. Starting with a word $a = 1\overline{111222222}$. Then we have the limit $P = \overline{000010101}$, so the word a represents the point $\frac{15}{365}$.

3-3. Some properties of the Tent map

In this section, we show that the decoding map is uniformly continuous, and prove that there exists a point $x \in I$ such that the orbit $\{f^n(x)\}$ is dense in I.

THEOREM 3.3.1. *Let n be a prime number. Then the number of words with period n is exactly $2^n - 2$. Therefore, the number of periodic points of prime period n under the tent map is $2^n - 2$.*

EXAMPLE 3. The number of words of period 3 is 6.

In fact,

$$a^1 = \overline{112} \quad a^2 = \overline{121} \quad a^3 = \overline{211}$$
$$a^4 = \overline{212} \quad a^5 = \overline{221} \quad a^6 = \overline{122}$$

Note that the words $a^1, a^2, a^3, a^4, a^5, a^6$ represent the numbers $\frac{14}{63}, \frac{28}{63}, \frac{56}{63}, \frac{6}{7}, \frac{4}{7}, \frac{2}{7}$ respectively.

Now we define a metric d on the sequence space S_2 by

$$d(a_1a_2a_3\cdots, b_1b_2b_3\cdots) = \sum \frac{|a_i - b_i|}{2^i},$$

where $a_1a_2a_3\cdots, b_1b_2b_3\cdots \in S_2$. Then d is a well-defined metric on S_2 and $G : S_2 \to S_2$ is continuous.

LEMMA 3.3.2. *The decoding map $D : S_2 \to I$ is uniformly continuous.*

Proof Let $\epsilon > 0$ be given. Take N such that $\sum_{N+1}^{\infty} \frac{1}{2^n} < \epsilon$. And let $\delta = 2^{-N}$. Now we consider two words $a = a_1a_2a_3\cdots$ and $b = .b_1b_2b_3\cdots$ in the sequence space S_2 on two symbols. Then we know that

$$d(a,b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k} < 2^{-N} = \delta.$$

Then $a_i = b_i$ for all $i \leq N$.

Since D is a decoding map, $D(a) = \sum_{n=1} \frac{x_n}{2^n}$ and $D(b) = \sum_{n=1} \frac{y_n}{2^n}$. Then we know that $x_n = y_n$ for $n \leq N$. Hence

$$\begin{aligned} |D(a) - D(b)| &= \left|\sum_{n=1} \frac{x_n}{2^n} - \sum_{n=1} \frac{y_n}{2^n}\right| \\ &= \left|\sum_{n=1} \frac{x_n - y_n}{2^n}\right| \\ &\leq \sum_{n=N+1} \frac{|x_n - y_n|}{2^n} \\ &\leq \sum_{n=N+1} \frac{1}{2^n} < \epsilon. \end{aligned}$$

Thus the decoding map D is uniformly continuous.

But the encoding map E is not continuous.

LEMMA 3.3.3. *Let $f : I \to I$ be the tent map and $x \in I$. If $\{G^n(E(x))\}$ is dense in S_2, then $f^n(x)$ is dense in I.*

Proof Suppose that $\{G^n(E(x))\}$ is dense in S_2 for some $x \in I$. Let $\epsilon > 0$. Since D is uniformly continuous, there exists a $\delta > 0$ such that $d(a,b) < \delta \Rightarrow d(D(a), D(b)) < \epsilon$. Let $y \in I$. Then by hypothesis, there exists n such that $d(G^n(E(x)), E(y)) < \delta$. By Theorem 3.1.1 and Theorem 3.2.1, there exist encoding map E and decoding map D such that $f = D \circ G \circ E$. By inductively, we know $f^n = D \circ G^n \circ E$, hence we have

$$d(f^n(x), y) = d\big(D(G^n(E(x))), D(E(y))\big) < \epsilon.$$

THEOREM 3.3.4. *There exists a point $x \in I$ such that the orbit $\{f^n(x)\}$ is dense in I.*

Proof First we will show that there is a word a such that $\{G^n(a)\}$ is dense in S_2. We take all strings of length 1, length 2, length $3, \cdots$, and concatenate them together. Then we obtain a word

$$a = 1 \setminus 2 \setminus 11 \setminus 12 \setminus 21 \setminus 22 \setminus 111 \setminus 112 \setminus 121 \setminus 122 \setminus 211 \setminus 212 \setminus 221 \setminus 222 \cdots .$$

We can easily show that for any finite sequence $a_1 a_2 a_3 \cdots a_n$ there exists a positive integer m such that $G^m(a) = a_1 a_2 a_3 \cdots a_n \cdots$.

Let $b = b_1 b_2 b_3 \cdots$ be a sequence in S_2 and $\epsilon > 0$ be given. Then we can take a positive integer k such that $\sum_{n=k+1} \frac{1}{2^n} < \epsilon$. Then there exists a positive integer m such that $G^m(a) = b_1 b_2 b_3 \cdots b_n a_{m+n+1} \cdots$ with $d(G^m(a), b) < \sum\limits_{n=k+1} \frac{1}{2^n} < \epsilon$. This means that $\{G^n(a)\}$ is dense in S_2. By Lemma 3.3.3, the orbit $\{f^n(D(a))\}$ is dense in I.

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