Journal of the Korea Society of Mathematical Education
Series A: Jun. 1994, Vol. 33, No. 1, 129-136

ON THE DISTRIBUTIVITY IN SUB(d)

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1. Introduction

Consider the class of monomorphism with common codomain d, for any object d in a topos, denoted by Sub(d).

For any f and g in Sub(d), defining an "inclusion" relation $f \subseteq g$, then (Sub(d), $\subseteq$) form a poset.

The following operations are defined on the Sub(d).

$$- : \mathrm{Sub(d)} \to \mathrm{Sub(d)}.$$
$$\cap : \mathrm{Sub(d)} \times \mathrm{Sub(d)} \to \mathrm{Sub(d)}.$$
$$\cup : \mathrm{Sub(d)} \times \mathrm{Sub(d)} \to \mathrm{Sub(d)}.$$
$$\Rightarrow : \mathrm{Sub(d)} \times \mathrm{Sub(d)} \to \mathrm{Sub(d)}.$$

For any f, g and h in Sub(d), $f \Rightarrow (g \cap h)$ is monic isomorphic to the distributivity.

But $f \cap (g \Rightarrow h)$ is not monic isomorphic to the distributivity, thus we show that $f \cap (g \Rightarrow h)$ is contained in it and the counter example of the reverse case is constructed.

In any topos, $f \Rightarrow g$ is contained in $-g \Rightarrow -f$. We show that $f \Rightarrow g$ is monic isomorphic to $-g \Rightarrow -f$ in Boolean topos.

Finally, we show $(g \cap h) \Rightarrow f$ contains the distributivity and the counter example is constructed for the reverse part.

2. Preliminaries

DEFINITION 2.1. *Given $f : a \to d$ and $g : b \to d$ in a category C we put $f \leq g$ if there is a C-arrow $h : a \to b$ such that $f = g \circ h$*

DEFINITION 2.2. *Given $f : a \to d$ and $g : b \to d$ in a category C, f and g are isomorphic subobjects if $f \leq g$, and $g \leq f$, denoted by $f \simeq g$*

DEFINITION 2.3. *Sub(d)=$\{[f] : f$ is a monic with cod $f = d\}$ where $f : a \to d$ determines an equivalence class*

$$[f] = \{g : f \simeq g\}.$$

DEFINITION 2.5. *An elementary topos is a category ε such that*

(1) *ε is finitely complete.*
(2) *ε is finitely co-complete.*
(3) *ε has exponentiation.*
(4) *ε has a subobject classifier.*

EXAMPLE 2.6. Let $M_2 = (2, \cdot, 1)$ where $2 = \{0, 1\}$ and is defined by $1 \cdot 1 = 1, 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$.
Then M_2 is a monoid with identity 1 in which 0 has no inverse.

The category of M_2-Sets is a topos

In $M_2, \Omega = (L_2, \omega)$ when L_2 is the left ideals of M_2 and the action $\omega : 2 \times L_2 \to L_2$ defined by $\omega(m, B) = \{n : n \in 2 \text{ and } n \cdot m \in B\}$

DEFINITION 2.7. *Given $f : a \to d$, the complement of f (relative to d) is the subobject $-f : -a \to d$ whose character is $\neg \circ \chi_f$, Thus $-f$ is defined to be the pullback of T along $\neg \circ \chi_f$, yielding $\chi_{-f} = \neg \circ \chi_f$*

DEFINITION 2.8. *The intersection of $f : a \to d$ and $g : b \to d$ is the subobject $f \cap g : a \cap b \to d$ obtained by pulling T back along $\chi_f \cap \chi_g = \cap \circ < \chi_f \cap \chi_g >$. Hence $\chi_{f \cap g} = \chi_f \cap \chi_g$.*

LEMMA 2.9. *In any topos ε, if $f : a \to d$ and $g : b \to d$ have pullback then $\alpha : c \to d$, where $\alpha = g \cdot f' = f \cdot g'$ has character $X_f \cap X_g$,*
Thus $\chi_\alpha = \chi_{f \cap g}$ so $\alpha \simeq f \cap g$ and there is a pullback of the form

$$\begin{array}{ccc} a \cap b & \longrightarrow & b \\ \downarrow & & \downarrow g \\ a & \xrightarrow[f]{} & d \end{array}$$

Proof REF

DEFINITION 2.10. *If ε is a topos with classifier $T : 1 \to \Omega$, $\cap : \Omega \times \Omega \to$ is the character in ε of the product arrow $< T, T >: 1 \to \Omega \times \Omega$. Hence*

$$\begin{array}{ccc} 1 & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle <T,T>}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega \times \Omega & \xrightarrow[\cap]{} & \Omega \end{array}$$

is a pullback.

DEFINITION 2.11. *If ε is a topos with classifier $T : 1 \to \Omega$, $\Rightarrow: \Omega \times \Omega \to \Omega$ is the character of $e : C \to \Omega \times \Omega$ where the latter is the equalizer of $\Omega \times \Omega \rightrightarrows \Omega, \cap$ being the conjunction truth arrow, and p_1 the first projection arrow of the product $\Omega \times \Omega$.*

LEMMA 2.12. *(Sub(d),$\subseteq$) is a bounded lattice with unit I_d and zero O_d*

Proof REF

LEMMA 2.13. *For $f : a \to d$, we have $(f \cap -f) \simeq O_d$*

Proof REF

DEFINITION 2.14. *A Boolean algebra is a complemented distributive lattice.*

DEFINITION 2.15. *A topos is called Boolean if for every ε-object d, (Sub(d), $\subseteq$) is a Boolean algebra.*

DEFINITION 2.16. *If $f : a \to d$ and $g : b \to d$ are subobject of d then $f \Rightarrow g : (a \Rightarrow b) \to d$ is the subobject obtained by pulling T back along $\chi_f \Rightarrow \chi_g = (\Rightarrow \circ < \chi_f, \chi_g >)$.*
Thus

$$\begin{array}{ccc} a \Rightarrow & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle f \Rightarrow g}\downarrow & & \downarrow{\scriptstyle T} \\ d & \xrightarrow[\chi_f \Rightarrow \chi_g]{} & \Omega \end{array}$$

is a pullback. i.e. $\chi_{f \Rightarrow g} = (\chi_f \Rightarrow \chi_g)$.

3. Main Parts

LEMMA 3.1. *In Sub(d), $h \subseteq f \cap g$ iff $h \subseteq f$ and $h \subseteq g$*

Proof Let $h \subseteq f \cap g$, where $f \cap g : a \cap b \to d$, $h : c \to d$, $f : a \to d$ and $g : b \to d$. By Definition, there exist $k : c \to a \cap b$ such that $(f \cap g) \circ k = h$.
Consider $\alpha \circ k$ and $\beta \circ k$ where $\alpha : a \cap b \to a$, $\beta : a \cap b \to b$, then $f \circ (\alpha \circ k) = h$ and $g \circ (\beta \circ k) = h$.

By a pullback. $f \cap g = f \circ \alpha = g \circ \beta$.
Thus $f \circ (\alpha \circ k) = (f \cap g) \circ k = h$, $g \circ (\beta \circ k) = (f \cap g) \circ k = h$.

Conversely let $h \subseteq f$ and $h \subseteq g$. By Definition, there exist m and n such that $f \circ m = h$ and $g \circ n = h$.

By Definition, there exist a morphism $q : c \to a \cap b$ such that $\alpha \circ q = m$ and $\beta \circ q = n$. Thus $h = (f \cap g) \circ q$, since $\beta \circ q = n$ and $g \circ n = h$ implies $h = g \circ (\beta \circ q)$, by pullback $h = (f \cap g) \circ q$.

LEMMA 3.2. *In Sub(d), we have $h \subseteq (f \Rightarrow g)$ iff $f \cap h \subseteq g$*

Proof REF

PROPOSTION 3.3. $\{(f \Rightarrow g) \cap (f \Rightarrow h)\} \simeq \{f \Rightarrow (g \cap h)\}$

Proof REF

THEOREM 3.4. *In Sub(d), (for any topos) $\{f \cap (g \Rightarrow h)\} \subseteq \{(f \cap g) \Rightarrow (f \cap h)\}$ hold but the converse is false.*

Proof We know that $\{g \cap (g \Rightarrow h)\} \subseteq h$, By the property of $\cap$, $[f \cap \{g \cap (g \Rightarrow h)\}] \subseteq (f \cap h)$, $\{(f \cap g) \cap (g \Rightarrow h)\} \subseteq (f \cap h)$ is equivalent to $[\{(f \cap g) \cap f\} \cap (g \Rightarrow h)] \subseteq (f \cap h)$.

By LEMMA, $\{f \cap (g \Rightarrow h\}) \subseteq \{(f \cap g) \Rightarrow (f \cap h\})$ The counter example is constructed.

Consider sub(Ω) in M_2 where $\Omega = (L_2, \odot)$ $\odot : 2 \times L_2 \to L_2$ defined by $\odot(m, B) = \{n : n \in 2 \text{ and } n \cdot m \in B\}$ and $L_2 = \{2, \{0\}, \emptyset\}$.

Let $f : (\{\{0\}, \emptyset\}, \odot |_{\{0\}}, \emptyset\}) \to (L_2, \odot), g : (\{\{0\}\}, \odot |_{\{0\}}\}) \to (L_2, \emptyset)$ and $h : (\{\emptyset\}, \odot |_{\{\emptyset\}}) \to (L_2, \odot)$ be an inclusion.

By definition of $\Rightarrow$

$$\begin{array}{ccc} (\{\emptyset,2\},\odot\mid_{\{\emptyset,2\}}) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle g\Rightarrow h}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_g\Rightarrow\chi_h]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset\},\odot\mid_{\{\emptyset\}}) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle f\cap(g\Rightarrow h}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_f\cap(\chi_{g\Rightarrow h})]{} & \Omega \end{array}$$

is a pullback.

Thus $f\cap(g\Rightarrow h):(\{\emptyset\},\odot\mid_{\{\emptyset\}})\to\Omega$

$$\begin{array}{ccc} (\{\emptyset\},\odot\mid_{\{\emptyset\}}) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle f\cap g}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_f\cap\chi_g]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset\},\odot\mid_{\{\emptyset\}}) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle f\cap h}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_f\cap\chi_h]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset,2\},\odot\,|_{\{\emptyset,2\}}) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle (f\cap g)\Rightarrow(f\cap h)}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_{h\cap g}\Rightarrow\chi_{f\cap h}]{} & \Omega \end{array}$$

is a pullback. Thus $(f \cap g) \Rightarrow (f \cap h) : (\{\emptyset, 2\}, \odot\,|_{\{\emptyset,2\}}) \to \Omega$.
We assume $\{f \cap (g \Rightarrow h)\} \supseteq \{(f \cap g) \Rightarrow (f \cap h)\}$, then there exist $k : (\{\emptyset, 2\}, \odot\,|_{\{\emptyset,2\}}) \to (\{\emptyset\}, \odot\,|_{\{\emptyset\}})$ such that $\{f \cap (g \Rightarrow h)\} \circ k = \{(f \cap g) \Rightarrow (f \cap h\})$. But this is a contradiction, since $f \cap (g \Rightarrow h)$ and $(f \cap g) \Rightarrow (f \cap h)$ are inclusion.

THEOREM 3.5. *In any Boolean topos, we have*

$$(f \Rightarrow g) \simeq (-g \Rightarrow -f)$$

Proof In any topos, we know that $(f \Rightarrow g) \subset (-g \Rightarrow -f)$. And it is also true that $(-g \Rightarrow -f) \subseteq [\{-(-f)\} \Rightarrow \{-(-g)\}]$
i.e. $(-g \Rightarrow -f) \subseteq (- - f \Rightarrow - - g)$.
By hypothesis, (Sub(d), $\subset$) is a Boolean algebra.
We have the fact that $f = - - f$ and $g = - - g$, REF[3rd].
Therefore $(f \Rightarrow g) \subseteq (-g \Rightarrow -f) \subseteq (f \Rightarrow g)$ is hold.
Thus $(f \Rightarrow g) \simeq (-g \Rightarrow -f)$

THEOREM 3.6. *In Sub(d) (for any topos),* $\{(g \cap h) \Rightarrow f\} \supseteq \{(g \Rightarrow f) \cap (h \Rightarrow f)\}$ *hold, but the converse is false.*

Proof It is always true that $(g \Rightarrow f) \subseteq (g \Rightarrow f)$
By Lemma, $\{g \cap (g \Rightarrow f)\} \subseteq f$. Similary $\{h \cap (h \Rightarrow f)\} \subseteq f$
By Lemma, $[\{g \cap (g \Rightarrow f)\} \cap \{h \cap (h \Rightarrow f)\}] \subseteq f$.
It is equivalent to $[(g \cap h) \cap \{(g \Rightarrow f) \cap (h \Rightarrow f)\}] \subseteq f$.
By Lemma, $\{(g \Rightarrow f) \cap (h \Rightarrow f)\} \subseteq \{(g \cap h) \Rightarrow f\}$.
The counter example of the converse is constructed.
Consider Sub(Ω) in M_2, where $\Omega = (L_2, \odot)\odot : 2 \times L_2 \to L_2$ defined by $\odot(m, B) = \{n : n \in 2 \text{ and } n \cdot n \in B\}$ and $L_2 = \{2, \{0\}, \emptyset\}$.
Let $g : (\{2, \{0\}\}, \odot\,|_{\{2,\{0\}\}}) \to (L_2, \odot), h : (\{\{0\}\}, \odot\,|_{\{\{0\}\}}) \to (L_2, \odot)$ and $f : (\{\{\emptyset\}\}, \odot\,|_{\{\{\emptyset\}\}}) \to (L_2, \odot)$ be an inclusion.
By Definition of $\cap$

$$\begin{array}{ccc} (\{\emptyset\}, \odot \mid_{\{\emptyset\}}) & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle g\cap h}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_g \cap \chi_h]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset, 2\}, \odot \mid_{\{\emptyset,2\}}) & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle (g\cap h)\Rightarrow f}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_{g\cap h} \Rightarrow \chi_f]{} & \Omega \end{array}$$

is a pullback.

Thus $(g \cap h) \Rightarrow f : (\{\emptyset, 2\}, \odot \mid_{\{\emptyset,2\}}) \to \Omega$

$$\begin{array}{ccc} (\{\emptyset\}, \odot \mid_{\{\emptyset\}}) & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle g\Rightarrow h}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_g \cap \chi_h]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset, 2\}, \odot \mid_{\{\emptyset,2\}}) & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle (h\Rightarrow f)}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_h \Rightarrow \chi_f]{} & \Omega \end{array}$$

is a pullback.

$$\begin{array}{ccc} (\{\emptyset\}, \odot\mid_{\{\emptyset\}}) & \xrightarrow{\quad !\quad} & 1 \\ {\scriptstyle (g\Rightarrow f)\cap(h\Rightarrow f)}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega & \xrightarrow[\chi_{g\Rightarrow f}\Rightarrow\chi_{h\Rightarrow f}]{} & \Omega \end{array}$$

is a pullback.

Thus $(g \Rightarrow f) \cap (h \Rightarrow f) : (\{\emptyset\}, \odot\mid_{\{\emptyset\}}) \to \Omega$.

We assume $\{(g \cap h) \Rightarrow f\} \subseteq \{(g \Rightarrow f) \cap (h \Rightarrow f)\}$, then there exist $u : \{\emptyset, 2\} \to \{\emptyset\}$ such that $(g \Rightarrow f) \cap (h \Rightarrow f) \circ u = \{(g \cap h) \Rightarrow f\}$. But this is a contradiction, since $(g \Rightarrow f) \cap (h \Rightarrow f)$ and $(g \cap h) \Rightarrow f$ are inclusion.

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