ON THE COMPLETENESS OF $\mathscr{L}F(X, Y)$

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ABSTRACT

Let X, Y be normed linear spaces, and let ρ_1, ρ_2 be lower semi-continuous fuzzy norms on X, Y respectively, and have the bounded supports on X, Y respectively. In this paper, we prove that if Y is complete, the set of all fuzzy continuous linear maps from X into Y is a fuzzy complete fuzzy normed linear space.

I. Introduction.

Since Katsaras and Liu [1] have introduced the notions of fuzzy vector spaces and fuzzy topological vector spaces, the theory of fuzzy topological vector spaces were developed by [2, 3, 4, 5]. In [3], Katsaras defined the fuzzy norm on a vector space and studied its properties. Krishna and Sarma [5] studied the properties of fuzzy norms on the set of all fuzzy continuous linear maps from a fuzzy normed linear space into another fuzzy normed linear space. In [6], Rhie, Choi and Kim introduced the notions of the fuzzy α-Cauchy sequence and the fuzzy completeness, and studied some of related properties of fuzzy normed linear spaces.

Let X, Y be normed linear spaces, and let ρ_1, ρ_2 be lower semi-continuous fuzzy norms on X, Y respectively, and have the bounded supports on X, Y respectively. In this paper, we prove that if Y is complete, the set of all fuzzy continuous linear maps from X into Y is a fuzzy complete fuzzy normed linear space.

II. Preliminaries.

Throughout this paper X is a vector space over the field $K(R$ *or* $C)$. Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the crisp set A. By a fuzzy point μ we mean a fuzzy subset $\mu : X \to [0, 1]$ such that

$$\mu(z) = \begin{cases} \alpha, & if \quad z = x \\ 0, & otherwise \end{cases}$$

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where $\alpha \in (0, 1)$, and I^X denoted the set $\{\mu \mid \mu: X \to [0, 1]\}$. We usually denote the fuzzy point with support x and value α by (x, α).

Definition 2.1 [5]. Let (X, τ) be a fuzzy topological space. A fuzzy subset μ in X is called a neighbourhood of (x, α) if there exists $\psi \in \tau$ with $\psi(x) \geq \alpha$ and $\psi \leq \mu$.

Definition 2.2 [5]. Let (X, τ) be a fuzzy topological space, $\{\mu_n, \alpha_n)\}$ a sequence of fuzzy points in X and μ a fuzzy point in X. We say that $\{\mu_n\}$ converges to μ, written as $\mu_n \to \mu$ if for every neighbourhood N of μ there exists a positive integer M such that $n \geq M$ implies $\mu_n \leq M$.

Definition 2.3 [3]. A fuzzy norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing and $inf_{t>0}\, t\rho(x) = 0$ for $x \neq 0$.

Theorem 2.4 [3, Theorem 4.2]. If ρ is a fuzzy norm on X, then the family $\mathcal{B}_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1, t > 0\}$ is a base at zero for a fuzzy linear topology τ_ρ.

Definition 2.5 [3]. Let ρ be a fuzzy norm on a linear space. The fuzzy topology τ_ρ in Theorem 2.4 is called the fuzzy topology induced by the fuzzy norm ρ. And a linear space equipped with a fuzzy norm is called a fuzzy normed linear space.

Definition 2.6 [4, 5]. If ρ is a fuzzy norm on X, P_ε is defined by $P_\varepsilon(x) = inf\{t > 0 \mid t\rho(x) > \varepsilon\}$ for each $\varepsilon \in (0, 1)$ and $P_\alpha: X \to R_+$ defined by $P_\alpha(x) = sup_{\varepsilon < \alpha} P_\varepsilon(x)$ for every $x \in X$.

Theorem 2.7 [4, Theorem 3.2]. P_ε is a norm on X for each $\varepsilon \in (0, 1)$ if and only if ρ is a fuzzy norm on X.

Theorem 2.8 [3, Theorem 4.7]. Let (X_1, ρ_1), (X_2, ρ_2) be fuzzy normed linear spaces and $f: X_1 \to X_2$ a linear map. Then, f is fuzzy continuous if and only if for each θ, $0 < \theta < 1$, there exists $t > 0$ such that $\theta \wedge \rho_1(tx) \leq \rho_2(f(x))$ for all $x \in X$.

Definition 2.9 [6]. Let $\alpha \in (0, 1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a fuzzy α-Cauchy sequence in a fuzzy normed linear space (X, ρ) if for each zero neighbourhood N with $N(0) > \alpha$, there exists a positive integer M such that $n, m \geq M$ implies $\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \leq N$.

Theorem 2.10 [6, Theorem 3.2] Let (X, ρ) be a fuzzy normed linear space and $\alpha \in (0, 1)$. Then $\{(x_n, \alpha_n)\}$ is a fuzzy α-Cauchy if and only if for each $t > 0$, there exists a positive integer M such that $n, m \geq M$ implies $\alpha_n \wedge \alpha_m \leq \alpha$ and $P_{(\alpha_n \wedge \alpha_m)} - (x_n - x_m) < t$.

Theorem 2.11 [5, Theorem 3.2]. Let (X, ρ) be a fuzzy normed linear space and $\alpha \in (0, 1)$. Then $\{\mu_n = (x_n, \alpha_n)\}$ converges to (x, α) if and only if for every $t > 0$, there exists a positive integer M such that $n \geq M$ implies $\alpha_n \leq \alpha$ and $P_{\alpha_n} - (x_n - x) < t$.

Definition 2.12 [6]. A fuzzy normed linear space (X, ρ) is said to be fuzzy α-complete if every fuzzy α-Cauchy sequence $\{\mu_n\}$ converges to a fuzzy point $\mu = (x, \alpha)$. (X, ρ) is said to be fuzzy complete if it is fuzzy

α-complete for every $\alpha \in (0, 1)$.

Ⅲ. Main Results

In this section, we deal with the completeness of the set of all fuzzy continuous linear maps from a fuzzy normed space into another fuzzy normed linear space.

Definition 3.1 [5]. Let (X, ρ_1), (Y, ρ_2) be fuzzy normed vector spaces and $\mathscr{L}F(X, Y)$ be the vector space of all fuzzy continuous linear maps from (X, ρ_1) to (Y, ρ_2). For each $\theta \in (0, 1)$, $t_\theta : \mathscr{L}F(X, Y) \to R_+$ is defined by $t_\theta(f) = inf\{s > 0 \mid \rho_2(f(x)) \geq \theta \wedge \rho_1(sx)$ for all $x \in X\}$.

We write $t_\theta(f) = t(\theta, f)$.

Definition 3.2 [5]. We define $\rho_* : \mathscr{L}F(X, Y) \to [0, 1]$ by $\rho_*(f) = sup_{\theta \in (0, 1)}(\theta \wedge 1/[t(\theta, f)])$ for $f \in \mathscr{L}F(X, Y)$.

Theorem 3.3 [5, Theorem 4.7]. ρ_* is a fuzzy norm on $\mathscr{L}F(X, Y)$.

Theorem 3.4 [5, Theorem 4.10]. Let $(X, \|\cdot\|_1)$, $(Y, \|\cdot\|_2)$ be normed linear spaces over the field K and $\rho_1 = \chi_{B_1}$ and $\rho_2 = \chi_{B_2}$, $B_1 = \{x \in X \mid \|x\|_1 \leq 1\}$ and $B_2 = \{y \in Y \mid \|y\|_2 \leq 1\}$, $\mathscr{L}F(X, Y)$ be as earlier and ρ_* be the fuzzy norm on $\mathscr{L}F(X, Y)$. Then $P^*_\varepsilon(f) = \varepsilon\|f\|$.

For a sepecial case of Theorem 4.9 of [5], one can easily get following Lemma.

Lemma 3.5 Let $(X, \|\cdot\|_1)$, $(Y, \|\cdot\|_2)$ be normed linear spaces over the field K and $\rho_1 = \chi_{B_1}$ and $\rho_2 = \chi_{B_2}$, $B_1 = \{x \in X \mid \|x\|_1 \leq 1\}$ and $B_2 = \{y \in Y \mid \|y\|_2 \leq 1\}$. If $f : (X, \rho_1) \to (X, \rho_2)$ is a fuzzy continuous linear map, then $f : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_2)$ is continuous.

Theorem 3.6 Let $(X, \|\cdot\|_1)$, $(Y, \|\cdot\|_2)$ be normed linear spaces over the field K and $\rho_1 = \chi_{B_1}$ and $\rho_2 = \chi_{B_2}$, $B_1 = \{x \in X \mid \|x\|_1 \leq 1\}$ and $B_2 = \{y \in Y \mid \|y\|_2 \leq 1\}$, $\mathscr{L}F(X, Y)$ be as earlier and ρ_* be the fuzzy norm on $\mathscr{L}F(X, Y)$.

If $(Y, \|\cdot\|_2)$ is complete, then $(\mathscr{L}F(X, Y), \rho_*)$ is fuzzy complete.

Proof. Fix $\alpha \in (0, 1)$ and let $\{(f_n, \alpha_n)\}$ be a fuzzy α-Cauchy sequence of fuzzy points in $(\mathscr{L}F(X, Y), \rho_*)$. For every $t > 0$, there exists a positive integer M such that $n, m \geq M$ implies $\alpha_n \wedge \alpha_m \leq \alpha$ and $P^*_{(\alpha_n \wedge \alpha_m)^-}(f_m - f_n) = sup_{\varepsilon < \alpha_n \wedge \alpha_m} P^*_\varepsilon(f_m - f_n) = sup_{\varepsilon < \alpha_n \wedge \alpha_m} \varepsilon\|f_m - f_n\| = (\alpha_m \wedge \alpha_n)\|f_m - f_n\| < t$. That is, $\{f_n\}$ is a crisp Cauchy sequence in $(B(X, Y), \|\cdot\|)$. Since $B(X, Y)$ is complete, there exists $f \in B(X, Y)$ such that $\|f_n - f\|$ converges to 0 and $\alpha_n\|f_n - f\|$ converges to 0. Therefore, for every $t > 0$, there exists a positive integer M such that for $n \geq M$, $\alpha_n \leq \alpha$ and $P^*_{\alpha_n^-}(f_n - f) = \alpha_n\|f_n - f\| < t$. Hence $\{(f_n, \alpha_n)\}$ converges to (f, α) with respect to the topology τ_{ρ_*}. Therefore $(\mathscr{L}F(X, Y), \rho_*)$ is fuzzy α-complete for each $\alpha \in (0, 1)$. This complete the proof of the theorem.

Definition 3.7 [3]. Two fuzzy norms ρ_1, ρ_2 on X are said to be equivalent if $\tau_{\rho_1} = \tau_{\rho_2}$.

Theorem 3.8 ([3] corollary 4.9). The fuzzy norms ρ_1, ρ_2 on a linear space X are equivalent if and only if for each $\theta \in (0, 1)$, there exists $t > 0$ such that $\theta \wedge \rho_1(tx) \leq \rho_2(x)$ and $\theta \wedge \rho_2(tx) \leq \rho_1(x)$ for all $x \in X$.

Theorem 3.9 Let ρ be a lower semi-continuous fuzzy norm on a normed linear space X and have the bounded support. Then it is equivalent to the fuzzy norm χ_B, where B is the closed unit ball of X.

Proof. Let $\theta \in (0, 1)$ be given. Since ρ is lower semicontinuous. $\rho_\theta = \{x \in X \mid \rho(x) > \theta\}$ is open. Thus there exists $t_1 > 0$ such that $t_1^{-1} B \subset \rho_\theta$.

Hence for every $x \in X$, $\theta \wedge \chi_B(t_1 x) = \theta \times \chi_B(t_1 x) \leq \rho(x)$.

Since ρ has the bounded support, there exists $t_2 > 0$ such that supp $\rho \subset t_2 B$ and so $\|x\| > t_2$ implies $\rho(x) = 0$. Then we have $\rho(t_2 x) \leq \chi_B(x)$.

Take $t_0 = \max(t_1, t_2)$. Then $\theta \wedge \chi_B(t_0 x) = \rho(x)$ and $\theta \wedge \rho(t_0 x) \leq \chi_B(x)$ for every $x \in X$. By the Theorem 3.8, the proof is completed.

Theorem 3.10 Let ρ_1, ρ_2 are lower semi-continuous fuzzy norms on normed linear spaces X, Y respectively and have the bounded supports on X, Y respectively. If $(Y, \|\cdot\|)$ is complete, then $(\mathscr{L} F(X, Y), \rho_*)$ is fuzzy complete.

Proof. By Theorem 3.9, ρ_1 is equivalent to the fuzzy norm χ_{B_1} and ρ_2 is equivalent to the fuzzy norm χ_{B_2}. Also, by Theorem 3.6, $(\mathscr{L} F(X, Y), \rho_*)$ is fuzzy complete.

Corollary 3.11. Let X be a normed linear space on K and ρ_1 a fuzzy norm on X. If ρ_1 is lower semi-continuous and has bounded support, then $\mathscr{L} F(X, Y)$ is fuzzy complete.

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