Fuzzy Hypergraph

Hyung Lee-Kwang*, Gil Rok Oh** and Choong Ho Cho***

ABSTRACT

In this paper, the hypergraph is fuzzified. In a hypergraph, there are 3 types of sets which can be fuzzified. According to the fuzzification level, 7 types of fuzzy hypergraphs can be obtained. After defining the 7 types of hypergraphs, some interesting concepts are developed such as the order, size and degree of such as the order, size and degree of the fuzzy hypergraph. The proposed fuzzy hypergraphs and

Ⅰ. Introduction

The hypergraph was introduced by Berge[10]-[11] and has been considered as a useful tool to analyze the structure of a system and to represent a partition, covering and clustering[10]-[12]. The notion of hypergraph has been extended to the fuzzy theory and the concept of fuzzy hypergraph was proposed by Kaufmann[1]. However, it has been pointed out the Kaufmann's definition of fuzzy hypergraph is not appropriate to represent various fuzzy systems.

In this paper, we generalize the concept of fuzzy hypergraph. In a hypergraph, there are 3 types of sets which can be fuzzified. According to the fuzzification level of the 3 types sets, we can obtain 7 types of fuzzy hypergraph. After defining the 7 types of fuzzy hypergraph, we develop some interesting concepts such as the order, size and degree of the fuzzy hypergraph. The proposed fuzzy hypergraphs and concepts can be used to represent and characterize various fuzzy systems.

In section 2, brief reviews on the hypergraph and fuzzy set are given. Section 3 introduces 7 types of fuzzy hypergraphs, and in section 4 we develop some interesting concepts of the fuzzy hypergraph.

Ⅱ. Hypergraph and Fuzzy Set

The hypergraph $H=(V, \varepsilon)$ was proposed by Berge[10] and is defined as follows;

$H=(V, \varepsilon)$ where

$V=\{x_1, x_2,\ldots,x_n\}$: finite set of vertices

*Associate Professor, Dept. Computer Science, KAIST, Taejon, 305-701, Seoul KOREA

**Vice-President, ETRI, Taejon, 305-350, Seoul KOREA

***Assistant Professor, Dept. Computer Science and Statistics, Univ. of Soonchunhyang, 336-600, Seoul KOREA

$\varepsilon = \{E_i \mid i \in I\}$: family of subsets of V

$E_i \neq \emptyset \ (i \in I)$

$\cup_{i \in I} E_i = V$

The set V is called as the set of vertices and ε is the set of edges(or hyperedges) E_i. In the graph, the edge E_i is represented by a continuous line surrounding its vertices if $|E_i| \geq 2$; if $|E_i| = 1$ by a cycle on the element. IF $|E_i| = 2$ for all i, the hypergraph becomes an ordinary graph.

In a hypergraph, two vertices x and y are adjacent if there exists an edge which contains the two vertices. Two edges E_i and E_j are adjacent if their intersection is not empty($E_i \cap E_j \neq \emptyset$, $i \neq j$).

In a hypergraph $H = (V, \varepsilon)$, some terminologies can be defined as follows:

1) the order of H: the number of vertices $|V|$

2) the size of H: the number of edges $|E|$

3) the order of edge E_i: the number of vertices in E_i, that is $|E_i|$

4) degree of vertex: the number of edges connecting the vertex

5) degree of edge: the number of adjacent edges

In the literature[2]-[8], when a fuzzy set $\widetilde{A}$ is given in a universal set X, the fuzzy set $\widetilde{A}$ is represented by:

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) \mid \mu_{\widetilde{A}}(x) > 0\}$$

where $\mu_{\widetilde{A}}(x)$ is the membership function of element x in the fuzzy set $\widetilde{A}$. The support Supp($\widetilde{A}$) of a fuzzy set $\widetilde{A}$ is a crisp set. We can cut a fuzzy set $\widetilde{A}$ at the level α and have α-cut set A_α of $\widetilde{A}$ such as:

$$A_\alpha = \{x \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$$

When two fuzzy set $\widetilde{A}$ and $\widetilde{B}$ are given, we can obtain the union $\widetilde{A} \cup \widetilde{B}$ and the intersection $\widetilde{A} \cap \widetilde{B}$ which are defined by their membership function $\mu_{\widetilde{A} \cup \widetilde{B}}(x)$ and $\mu_{\widetilde{A} \cap \widetilde{B}}(x)$ respectively:

$$\mu_{\widetilde{A} \cup \widetilde{B}}(x) = \max [\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)]$$

$$\mu_{\widetilde{A} \cap \widetilde{B}}(x) = \min [\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)]$$

Ⅲ. Fuzzy Hypergraph

In the hypergaph, there are 3 types of sets: the set vertices V, the set of edges ε and the edges(set of vertices) E_i. By defining the three types of sets, we can obtain 7 types of fuzzy hypergraphs. In the following, we will define the 7 types of fuzzy hypergraphs.

Definition: type-1 fuzzy hypergraph

In this hypergraph, the sets V and ε are crisp and the sets E_i are fuzzy as follows:

$H = (V, \varepsilon)$

$V = \{x_1, x_2, \ldots, x_n\}$

$\varepsilon = \{\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_j, \ldots, \widetilde{E}_m\}$

$\widetilde{E}_i = \{(x_j, \mu_i(x_j))\}$

$\mu_i(x_j)$ the membership degree of x_j in the edge $\widetilde{E}_j$

$\widetilde{E}_i \neq \phi$, $i = 1, \ldots, m$

$\cup_i \text{Supp}(\tilde{E}_i) = V$

This type of fuzzy hypergraph is similar to the fuzzy hypergraph defined by Kaufmann. In his hypergraph, the following condition is added.

$\cup_i \tilde{E}_i = V$

But in this paper this condition is eliminated because it limits the modelling power of the fuzzy hypergraph.

Definition: type-2 fuzzy hypergraph

The sets V and E_j are crisp and $\tilde{\varepsilon}$ is fuzzy.

$H = (V, \tilde{\varepsilon})$

$V = \{x_1, x_2, \ldots, x_n\}$

$E_i = \{x_j \mid x_j \in E_i\}$

$\tilde{\varepsilon} = \{(E_i, \mu_{\tilde{\varepsilon}}(E_i))\}$

$\mu_{\tilde{\varepsilon}}(E_i)$: the membership of E_i in $\tilde{\varepsilon}$

$E_i \neq \emptyset$

$\cup_i E_i = V$

Definition: type-3 fuzzy hypergraph

The set V is crisp, and the sets $\tilde{\varepsilon}$ and $\tilde{E}_i$ are fuzzy.

$H = (V, \tilde{\varepsilon})$

$V = \{x_1, x_2, \ldots, x_n\}$

$\tilde{\varepsilon} = \{(\tilde{E}_i, \mu_{\tilde{\varepsilon}}(\tilde{E}_i))\}$

$\tilde{E}_i = \{(x_j, \mu_i(x_j))\}$

$\mu_{\tilde{\varepsilon}}(\tilde{E}_i)$: the membership of $\tilde{E}_i$ in $\tilde{\varepsilon}$

$\mu_i(x_j)$: the membership of x_j in $\tilde{E}_i$

$\tilde{E}_i \neq \emptyset$

$\cup_i \text{Supp}(\tilde{E}_i) = V$

In this case, the set $\tilde{\varepsilon}$ is a level-two fuzzy set.

Definition: type-4 fuzzy hypergraph

The set $\tilde{V}$ is fuzzy, and the sets ε and E_i are crisp.

$H = (\tilde{V}, \varepsilon)$

$\tilde{V} = \{(x_i, \mu_{\tilde{V}}(x_i)\}$

$\mu_{\tilde{V}}(x_i)$: the membership of x_i in $\tilde{V}$

$\varepsilon = \{E_1, E_2, \ldots, E_m\}$

$E_i = \{x_j \mid x_j \in E_i\}$

$E_i \neq \emptyset, i = 1, \ldots, m$

$\cup_i E_i = \mathrm{Supp}(\widetilde{V})$

Definition: type-5 fuzzy hypergraph

The set ε is crisp, and the sets $\widetilde{V}$ and $\widetilde{E}_i$ are fuzzy.

$H = (\widetilde{V}, \varepsilon)$

$\widetilde{V} = \{(x_i, \mu_{\widetilde{V}}(x_i))\}$

$\mu_{\widetilde{V}}(x_i)$: the membership of x_i in $\widetilde{V}$

$\varepsilon = \{\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_m\}$

$\widetilde{E}_i = \{(x_j, \mu_i(x_j))\}\ i = 1, 2, \ldots, m$

$\mu_i(x_j)$: the membership of x_j in $\widetilde{E}_i$

$\widetilde{E}_i \neq \emptyset\ i = 1, 2, \ldots, m$

$\cup_i \mathrm{Supp}(\widetilde{E}_i) = \mathrm{Supp}(\widetilde{V})$

Definition: type-6 fuzzy hypergraph

The set E_i is crisp, and the sets $\widetilde{V}$ and $\widetilde{\varepsilon}$ are fuzzy.

$H = (\widetilde{V}, \widetilde{\varepsilon})$

$\widetilde{V} = \{(x_i, \mu_{\widetilde{V}}(x_i))\}$

$\mu_{\widetilde{V}}(x_i)$: the membership of x_i in $\widetilde{V}$

$\widetilde{\varepsilon} = \{(\widetilde{E}_i, \mu_{\widetilde{\varepsilon}}(E_i))\}$

$\mu_{\widetilde{\varepsilon}}(E_i)$: the membership of E_i in $\widetilde{\varepsilon}$

$E_i \neq \emptyset\ i = 1, 2, \ldots, m$

$\cup_i E_i = \mathrm{Supp}(\widetilde{V})$

Definition: type-7 fuzzy hypergraph

The all the sets $\widetilde{V}$, $\widetilde{\varepsilon}$, $\widetilde{E}_i$ are fuzzy.

$H = (\widetilde{V}, \widetilde{\varepsilon})$

$\widetilde{V} = \{(x_i, \mu_{\widetilde{V}}(x_i))\}$

$\mu_{\widetilde{V}}(x_i)$: the membership of x_i in $\widetilde{V}$

$\widetilde{\varepsilon} = \{(\widetilde{E}_i, \mu_{\widetilde{\varepsilon}}(E_i))\}$

$\mu_{\widetilde{\varepsilon}}(E_i)$: the membership of E_i in $\widetilde{\varepsilon}$

$\widetilde{E}_i = \{(x_j, \mu_i(x_j))\}\quad i = 1, 2, \ldots, m$

$\mu_i(x_j)$: the membership of x_j in $\tilde{E}_i$

$\tilde{E}_i \neq \phi \; i=1, 2, \ldots, m$

$\cup_i E_i = \mathrm{Supp}(\tilde{V})$

Now we summarize the above 7 types of fuzzy hypergraph and the ordinary hypergraph as shwon in the Table 1. In the table, the number indicates the type of fuzzy hypergraphs, ahd the number 0 indicates the ordinary hypergraph. For example, for the type-4 graph, the number 4 is placed in the boxes fuzzy V, crisp ε and crisp E_i.

Table 1.

	V	ε	E_i
crisp	0123	01 45	02 46
fuzzy	4567	23 67	13 57

Ⅳ. Some Characteristics in Fuzzy Hypergraph

We can generalize the terminologies of the ordinary hypergraph to the fuzzy hypergraph. The order, size and degree in ordinary graph can be considered as the cardinality of the sets. Similarly, we apply the cardinality to the fuzzy set $\tilde{V}$, $\tilde{\varepsilon}$, and $\tilde{E}_i$ as follows:

1) order of H: $|\tilde{V}|$, the cardinality of $\tilde{V}$
2) size: $|\tilde{\varepsilon}|$, the cardinality of $\tilde{\varepsilon}$
3) order of edge: $|\tilde{E}_i|$, the cardinality of $\tilde{E}_i$
4) degree of vertex x_j: the sum of $\mu_i(x_j) = \sum_i \mu_i(x_j)$
5) degree of edge $\tilde{E}_i$: $\sum_k \mu_{\tilde{\varepsilon}}(\tilde{E}_k)$
   for x_j s.t. $\mu_{\tilde{E}_i}(x_j) > 0$ and $\mu_{\tilde{E}_k}(x_j) > 0$
6) adjacent vertices(x, y) at level α
   $\alpha = \max_i \min[\mu_i(x), \mu_i(y)]$
7) adjacent edges (E_j, E_k) at level β
   $\beta = \max_x \min[\mu_j(x), \mu_k(x)], \mu_j(x) > 0$

All the sets are noted by fuzzy sets such as $\tilde{V}$, $\tilde{\varepsilon}$ and $\tilde{E}_i$. These notations can be used for the crisp sets V, ε and E_i.

Ⅴ. Conclusion

The hypergraph is considered as a useful tool for system analysis and clustering. We have fuzzified the hypergraph and seen that we can obtain 7 types of fuzzy hypergraphs. It is natural to think that the various fuzzy hypergraphs can represent various fuzzified systems such as fuzzy clustering and fuzzy structured systems.

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