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Estimator of the Mean Residual Life for Some Parametric Families[1)]

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Abstract

In this paper we consider a new estimator of mean residual life(MRL), based on the partial moment of the distribution.The parameters of a partial moment are estimated by its maximum likelihood estimators when the underlying distribution is known. Though the new estimator is not a consistent estimator of the MRL,it is shown to have smaller mean squared error than the well known empirical MRL estimator for certain parametric families. Numerical summaries of the mean squared errors of the new estimator are presented.

1. Introduction

Let X be a non-negative random variable with right continuous distribution function F, and survival function $\overline{F} \equiv 1-F$. For example, X might represent the time to failure of a certain unit or the survival time of a patient after being diagnosed as having a certain type of disease. We assume that $F(0)=0$ and mean $\mu=E(X)=\int_0^\infty \overline{F}(x)dx$ is finite.

The mean residual life(MRL) function or remaining life expectancy funtion at age t is defined as

$$m(t) = E(X-t \mid X > t) = \begin{cases} \int_t^\infty \overline{F}(x)dx/\overline{F}(t) & \text{if} \quad \overline{F}(t)>0 \\ 0 & \text{if} \quad \overline{F}(t)=0, \end{cases} \tag{1.1}$$

In reliability theory the MRL function arises natually and is of practical interest in many applications, for example,survivorship studies in medical settings,life expectancy studies for life insurance and industrial burn-in procedures. The objective of burn-in procedure is to screen out defective items at early stages and thus improve the lifetimes of remaining survival items. One useful tool for analyzing 'burn-in' is to model the aging process of an item by the MRL function. The hazard rate function is also widely used for this

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purpose.Guess and Proschan(1988) and Park(1992) provided excellent references on the properties of MRL function and its applications.

In this paper we propose new parametric estimators of the MRL function when the underlying life distribution belongs to certain parametric families,such as gamma family,Weibull family,etc. Our approach for estimating the MRL function is based on the partial moment estimator, proposed recently by Park(1990). When the underlying distribution is known,the parameters in the expression of the partial moment approximation of MRL function are replaced by its maximum likelihood estimators (MLE) and as a result,new parametric estimators of the MRL function can be obtained. In Section 2, we give the partial moment estimator and the empirical estimator of MRL function for completeness of our discussions and we present the new estimator of the MRL function,using the MLE's of the parameters. In Section 3, we compare the MRL function with its partial moment approximation for various choices of parameters of gamma and Weibull families and present the parametric estimators of their MRL functions. In Section 4, the new estimators are compared numerically with the empirical MRL estimator by Monte Carlo simulation studies.We also give some comments on the estimator and discuss the situations in which our estimator is useful.

2. Estimator of MRL function

For completeness of our discussion, we present two known estimators of the MRL function. The empirical MRL estimator,denoted by $\widehat{m}_e(t)$, is obtained by replacing $\overline{F}$ of MRL function of (1.1) by its consistent empirical estimator as

$$\widehat{m}_e(t)=\begin{cases} (n-k)^{-1}\sum_{i=k+1}^{n}(X_{(i)}-t) & \text{for} \quad X_{(k)}\le t<X_{(k+1)} \\ 0 & \text{for} \quad t\ge X_{(n)}, \end{cases} \tag{2.1}$$

for $k=0,1,2,\cdots,n-1$; $X_{(0)}\equiv 0$ and $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ is the corresponding order statistics of a random sample $X_1, X_2, \cdots, X_n$ from F. Throughout we assume no ties. Under the assumption of continuity on F, ties will not occur with probability 1 and the estimator (2.1) is appropriate. In industrial or medical setting, ties may occur due to grouping of the data. For the case of ties, the estimator is slightly modified as given in Guess and Proschan (1988).

Yang (1978) has shown that $\widehat{m}_e(t)$ is not unbiased estimator of $m(t)$. More precisely, $E(\widehat{m}_e(t))=m(t)\cdot p$ (at least one $X_i>t$). Thus, in general the bias which is $E(\widehat{m}_e(t))-m(t)$ is negative. However, Yang (1978) proved that $\widehat{m}_e(t)$ is strongly consistent as the sample size increases to infinity. Hall and Wellner (1979) showed that var $(\sqrt{n}(\widehat{m}_e(t)-m(t))) \rightarrow \sigma^2(t)/\overline{F}(t)$, where $\sigma^2=\mathrm{var}(X-t|X>t)$ and thus

$$MSE(\widehat{m}_e(t)) \rightarrow 0 \text{ as } n\rightarrow\infty.$$

The MRL function can be rewritten as

$$\begin{aligned} m(t) &= \overline{F}^{-1}(t)\int_t^{\infty} x\, dF(x)-t \\ &= \overline{F}^{-1}(t)\left\{\mu-\int_0^t x\, dF(x)\right\}-t \\ &= (1-p)^{-1}\left\{\mu-\int_0^t x\, dF(x)\right\}-t, \end{aligned}$$

where $p=F(t)=p(X\le t)$ and $\int_0^t x\, dF(x)$ is defined as the first partial moment of X about the origin over (0, t) for fixed t. (cf. Choobinch and Branting, 1986) The partial moment approximation of $m(t)$ is given by

$$m_p(t)=\mu+\left[\frac{p}{1-p}\right]^{1/2}\sigma-t, \tag{2.2}$$

where μ and σ^2 are the mean and variance of X, respectively. Utilizing (2.2), Park (1990) proposed a new nonparametric estimator of $m(t)$ as

$$\widetilde{m}_p(t)=\begin{cases} \overline{X}+\left[\frac{\hat{p}}{1-\hat{p}}\right]^{1/2}S-t, & \text{for } t<X(n) \\ 0 & \text{for } t\ge X(n), \end{cases} \tag{2.3}$$

where $X_{(n)}=\max_{i\le j\le n} X_j$, $\overline{X}$, S^2 are sample mean and variance, respectively and $\hat{p}=n^{-1}\sum_{i=1}^{n} I(X_i\le t)$ is the proportion of observations that are less than or equal to t and $I(\cdot)$is an indicator function. Although $\widetilde{m}_p(t)$ is not a consistent estimator of $m(t)$, it is shown numerically that $\widetilde{m}_p(t)$ performs better than $\widehat{m}_e(t)$ for small sample size.

For our purpose of estimating the MRL function, we consider a parametric family whose probability density function is denoted by $f(x;\ )$, where is a vector of parameters. Let $\widehat{\ }$ be the maximum likelihood estimator of if $\widehat{\ }$ exists and let $\hat{\mu}=\hat{\mu}(\widehat{\ })$ and $\hat{\sigma}^2=\hat{\sigma}^2(\widehat{\ })$ be the maximum likelihood estimators μ of σ^2 and, respectively. By substituting $\hat{\mu}$ and $\hat{\sigma}$ for μ and σ in the expression of $m_p(t)$ in (2.2), we obtain a new parametric estimator, $\widehat{m_p}(t)$ as

$$\widehat{m_p}(t) = \hat{\mu} + \left[\frac{\hat{p}}{1-\hat{p}} \right]^{1/2} \hat{\sigma} - t, \qquad \text{for} \quad 0 \le t < \infty, \tag{2.4}$$

where $\hat{p} = \hat{F} = \int_0^t f(x;\ \widehat{\ })dx$ is the maximum likelihood estimator of $F(t)$. In the rest of this paper, $\widehat{m_p}(t)$ is referred to as the MLE of $m(t)$. The MLE is not a consistent estimator of $m(t)$, but the estimator is somewhat smoothing out the empirical estimator $\widehat{m_e}(t)$. This fact suggests that the mean squared error of $\widehat{m_p}(t)$ is likely to be smaller than that of $\widehat{m_e}(t)$, especially when the sample size n is small.

3. Parametric estimators of gamma and Weibull families

To study the performance of the MLE estimator, we first investigate the behavior of the partial moment approximation (2.2) of MRL function for several choices of parameters for gamma and Weibull families.

a) Gamma distribution :

$$\begin{aligned} & f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \mathrm{Exp}[-(x/\beta)], \quad \alpha>0, \beta>0, x\ge 0 \\ & m(t) = \int_t^{\infty}\int_x^{\infty} f(u;\alpha,\beta)\,du\,dx \ / \int_t^{\infty} f(u;\alpha,\beta)\,du \\ & \mu = \alpha\beta, \ \sigma^2 = \alpha\beta^2 \end{aligned} \tag{3.1}$$

b) Weibull distribution :

$$\begin{aligned} & f(x;c,\beta) = c(x/\beta)^{c-1}\beta^{-1}\mathrm{Exp}[-(x/\beta)^c], \quad c>0, \beta>0, x\ge 0 \\ & m(t) = \mathrm{Exp}[-(t/\beta)^c] \int_t^{\infty} \mathrm{Exp}[-(x/\beta)^c]dx \\ & \mu = \beta\Gamma(c^{-1}+1), \ \sigma^2 = \beta^2\{\Gamma(2c^{-1}+1) - [\Gamma(c^{-1}+1)]^2\}. \end{aligned} \tag{3.2}$$

For studying the accuracy of the approximation (2.2), we take $\beta = 1$ for both gamma and Weibull distributions. Tables 1 and 2 show numerical comparison of $m(t)$ and $m_p(t)$ of (2.2) for various choices of t and α for gamma distribution and for t and c for Weibull distribution, respectively.

Table 1. Numerical comparision of $m(t)$ and $m_p(t)$ for gamma distribution with $\beta=1$.

α	t	m(t)	$m_p(t)$	α	t	m(t)	$m_p(t)$
5	0.4	4.6003	4.6175	15	1.5	13.5000	13.5000
	0.8	4.2061	4.2841		3.0	12.0000	12.0032
	1.2	3.8315	3.9976		4.5	10.5008	10.5532
	1.6	3.4904	3.7483		6.0	9.0134	9.1450
	2.0	3.1905	3.5272		7.5	7.5857	7.8943
10	0.5	9.5000	9.5004	20	1.0	19.0000	19.0000
	1.5	8.5000	8.5046		3.0	17.0000	17.0000
	2.5	7.5022	7.5527		5.0	15.0000	15.0026
	3.5	6.5230	6.6824		7.0	13.0006	13.0298
	4.5	5.5061	5.9170		9.0	11.0123	11.1454

Table 2. Numerical comparision of $m(t)$ and $m_p(t)$ for Weibull distribution with $\beta=1$

	C=3		C=5		C=10		C=20	
t	m(t)	$m_p(t)$	m(t)	$m_p(t)$	m(t)	$m_p(t)$	m(t)	$m_p(t)$
0.1	.7838	.8032	.8182	.8188	.8514	.8514	.8735	.8737
0.3	.6113	.6467	.6198	.6285	.6514	.6516	.6735	.6735
0.5	.4624	.5114	.4341	.4556	.4518	.4549	.4735	.4736
0.7	.3489	.4006	.2803	.3081	.2604	.2707	.2737	.2752
0.9	.2652	.3292	.1700	.2068	.1098	.1253	.0887	.0952

The comparision shows that the approximation $m_p(t)$ is very good for smaller values of t, regardless of the values of α or C. The values of t listed in Table 1 are somewhat different depending on the values of α.

The reason is that we list only those values of t for which $m(t)$ is not quite equal or very close to $m_p(t)$. For each α,the values of $m(t)$ and $m_p(t)$ are almost equal when t is less than the smallest values of t given in Table 1.

However, for fixed t, the approximation gets better as α and C increase. It implies that the approximation works best when the underlying distribution is symmetric and t is relatively small. Note that for gamma family, as α increases the gamma distribution becomes symmetric. It is apparent from the tables that the approximation is not very good for the values of t beyond a certain bound, probably the mean or the median. It is also noted that $m(t)$ decreases as t increases to infinity, which is not surprising since the gamma distribution and Weibull distribution are decreasing MRL for $\alpha \geq 1$ and $C \geq 1$, respectively.

The MLE's of α, c, β for both families are straightforward to obtain. For gamma distribution of (3.1), the MLE $\tilde{\alpha}$ is the solution of the equation

$$n^{-1}\sum_{j=1}^{n}\log X_j - \log\overline{X} = (\Gamma'(\tilde{\alpha})/\Gamma(\tilde{\alpha})) - \log\tilde{\alpha}.$$

Instead of $\tilde{\alpha}$, for simplicity of computation we use its approximation

$$\hat{\alpha} = \frac{1}{4}Y^{-1}[1+(1+\frac{4}{3}Y)^{1/2}], \tag{3.3}$$

where $Y = \log\overline{X} - n^{-1}\sum_{j=1}^{n}\log X_i$ and $\hat{\beta} = \overline{X}/\hat{\alpha}$. The approximation (3.3) is suggested by Thom (1968). Thus, for gamma family $m(t)$ is estmated by

$$\widehat{m_p}(t) = \overline{X} + \left[\frac{\hat{p}\hat{\alpha}}{1-\hat{p}}\right]^{1/2}\hat{\beta} - t, \tag{3.4}$$

where $\hat{p} = \int_0^t f(x;\hat{\alpha},\hat{\beta})dx$.

For Weibull distribution of (3.2), the MLE's of C and β are obtained as

$$\hat{\beta} = [n^{-1}\sum_{i=1}^{n}X_i^{\hat{c}}]^{1/\hat{c}}$$

and

$$\hat{c} = [\,(\sum_{i=1}^{n}X_i^{\hat{c}}\log X_i)(\sum_{i=1}^{n}X_i^{\hat{c}})^{-1} - n^{-1}\sum_{i=1}^{n}\log X_i\,]^{-1}.$$

Thus, for Weibull family $m(t)$ is estimated by

$$\widehat{m_p}(t) = \hat{\mu} + [(1-e^{-(t/\hat{\beta})^{\hat{c}}})e^{(t/\hat{\beta})^{\hat{c}}}]^{1/2}\,\hat{\sigma} - t, \tag{3.5}$$

where $\hat{\mu} = \hat{\beta}\Gamma(\hat{c}^{-1}+1)$ and $\hat{\sigma} = \hat{\beta}\{\Gamma(2\hat{c}^{-1}+1) - [\Gamma(\hat{c}^{-1}+1)]^2\}^{1/2}$.

It is noted that both estimators (3.4) and (3.5) are defined for $t \in [0, \infty)$, which is not the case for either the empirical estimator in (2.1) or the partial moment estimator in (2.3).

4. Numerical comparision and conclusion

In this section we carried out some simulations for gamma distribution and Weibull distribution to see how our parametric estimators (3.4) and (3.5) perform with respect to its mean squared error (MSE). As a competitor, we use the empirical MRL estimator given in (2.1). Let

$$Se = E[\widehat{m_e}(t) - m(t)]^2$$

and

$$Sp = E[\widehat{m_p}(t) - m(t)]^2$$

be the MSE's of the empirical estimator $\widehat{m_e}(t)$ and the MLE $\widehat{m_p}(t)$, respectively. Tables 3 and 4 show the values of Se and Sp for several combinations of α, t and n for gamma distribution and for several combinations of c, t and n for Weibull distribution. For both distibutions, we assume the scale parameter β being equal to 1. The values in these tables are based on 2000 replications for each combination.

Table 3 shows that for gamma distribution, Sp is sligthly smaller than Se except when $\alpha=5$. It also shows that as α increases (that is, the distribution becomes more symmetric), the MLE $\widehat{m_p}(t)$ performs better than the empirical estimator $\widehat{m_e}(t)$. Table 4 shows that for Weibull distribution, the MSE of $\widehat{m_p}(t)$ is significantly smaller than that of $\widehat{m_e}(t)$ for all cases. It is also clear that in both cases, the MSE converges to 0 as $n \to \infty$ for fixed t.

We do not provide the values of bias of these estimators in this paper, but the simulation results show that the bias of $\widehat{m_e}(t)$ is smaller than the bias of $\widehat{m_p}(t)$ in most cases. The extended tables of bias and MSE can be provided by authors on request. This implies that the variance of $\widehat{m_e}(t)$ is much larger than the variance of $\widehat{m_p}(t)$. This fact is expected because when the sample size is small, it is more likely to have no observations at all beyond the value of t, at which $\widehat{m_e}(t)$ is defined to be 0 and thus $\widehat{m_e}(t)$ would have a greater variance.

Table 3. Numerical values of the MSE's of $\widehat{m_{e(t)}}$ and $\widehat{m_p}(t)$

for gamma distribution $f(x; \alpha, \beta=1)$

		α = 5				α = 10			
t		0.5	1.0	1.5	2.0	1.0	2.0	3.0	4.0
m(t)		4.5008	4.0154	3.5719	3.1905	9.0000	8.0004	7.0081	6.0534
n=3	Se	1.6318	1.6390	1.6337	1.6508	3.4474	3.4509	3.4371	3.3685
	Sp	1.6434	1.6712	1.7078	1.7354	3.4397	3.4292	3.4260	3.4329
n=5	Se	1.0688	1.0647	1.0689	1.0629	2.1204	2.1181	2.1061	2.0643
	Sp	1.0812	1.1103	1.1563	1.2029	2.1174	2.1112	2.1138	2.1400
n=7	Se	.7146	.7146	.7109	.7069	1.4083	1.4107	1.4015	1.3742
	Sp	.7152	.7306	.7664	.8101	1.4056	1.3976	1.3950	1.4178
n=10	Se	.4908	.4896	.4779	.4772	.9261	.9263	.9189	.9042
	Sp	.4898	.5039	.5412	.5888	.9233	.9137	.9109	.9418
n=20	Se	.2575	.2568	.2498	.2484	.5016	.5017	.5005	.4955
	Sp	.2571	.2707	.3105	.3640	.5012	.4990	.5053	.5486

		α = 15				α = 20			
t		.05	2.0	3.5	5.0	1.0	3.0	5.0	7.0
m(t)		14.5000	13.000	11.5000	10.0024	19.0000	17.0000	15.0000	13.0006
n=3	Se	5.1208	5.1208	5.1208	5.1711	6.6819	6.6819	6.6819	6.6819
	Sp	5.1224	5.1256	5.1108	5.0967	6.6822	6.6823	6.6728	6.6645
n=5	Se	3.1352	3.1352	3.1352	3.1352	4.0389	4.0389	4.0389	4.0326
	Sp	3.1350	3.1299	3.1090	3.0695	4.0389	4.0347	4.1048	3.9661
n=7	Se	2.0171	2.0171	2.0171	2.0102	2.6373	2.6373	2.6373	2.6374
	Sp	2.0170	2.0135	1.9979	1.9712	2.6373	2.6361	2.6281	2.6023
n=10	Se	1.5239	1.5239	1.5239	1.5264	2.1156	2.1156	2.1156	2.1121
	Sp	1.5239	1.5223	1.5122	1.4915	2.1156	2.1147	2.1071	2.0807
n=20	Se	.7414	.7414	.7420	.7433	.9748	.9748	.9748	.9744
	Sp	.7414	.7410	.7377	.7332	.9748	.9747	.9724	.9626

Table 4. Numerical values of the MSE's of $\widehat{m_e}(t)$ and $\widehat{m_p}(t)$ for Weibull distribution f(x;c, β=1)

t	c = 10				c = 15			
	.1	.3	.5	.7	.1	.3	.5	.7
m(t)	.8514	.6514	.4518	.2604	.8657	.6657	.4657	.2671
n=3 Se	.00440	.00440	.00427	.00378	.00128	.00128	.00128	.00121
Sp	.00030	.00017	.00013	.00023	.00011	.00010	.00008	.00007
n=5 Se	.00254	.00254	.00249	.00209	.00128	.00128	.00128	.00121
Sp	.00019	.00017	.00013	.00023	.00011	.00010	.00008	.00007
n=7 Se	.00187	.00187	.00187	.00155	.00089	.00089	.00089	.00083
Sp	.00013	.00011	.00009	.00019	.00008	.00008	.00006	.00005
n=10 Se	.00125	.00125	.00124	.00102	.00064	.00064	.00064	.00059
Sp	.00009	.00009	.00007	.00016	.00005	.00003	.00002	.00003
n=20 Se	.00062	.00062	.00061	.00052	.00032	.00032	.00032	.00030
Sp	.00004	.00004	.00003	.00013	.00003	.00003	.00002	.00003

t	c = 20				c = 25			
	.1	.3	.5	.7	.1	.3	.5	.7
m(t)	.8735	.6735	.4735	.2737	.8784	.6784	.4784	.2785
n=3 Se	.00117	.00117	.00117	.00113	.00079	.00079	.00079	.00078
Sp	.00011	.00011	.00010	.00006	.00008	.00008	.00007	.00005
n=5 Se	.00077	.00077	.00077	.00076	.00046	.00046	.00046	.00046
Sp	.00007	.00007	.00006	.00004	.00005	.00005	.00004	.00003
n=7 Se	.00049	.00049	.00049	.00049	.00035	.00035	.00035	.00034
Sp	.00005	.00005	.00004	.00003	.00003	.00003	.00003	.00003
n=10 Se	.00039	.00039	.00039	.00038	.00024	.00024	.00024	.00024
Sp	.00003	.00003	.00003	.0002	.00002	.00002	.00002	.00002
n=20 Se	.00019	.00019	.00019	.00018	.00012	.00012	.00012	.00012
Sp	.00002	.00002	.00002	.00001	.00001	.00001	.00001	.00001

Since the MLE $\widehat{m_p}(t)$ is based on the approximation $m_p(t)$, it is not a consistent estimator of $m(t)$. However, as Tables 1 and 2 show, the approximation is quite good for both gamma and Weibull families when t is small. Thus, the MLE would perform well when t is small and n is relatively small. Especially, when the underlying distribution is known to have either a gamma distribution with large shape parameter or a Weibull distribution and when only a small sample is available, then the MLE is recommended. Another advantage of the MLE is that the estimator is a smooth curve, rather than a step function. Thus, although the MLE may not be very good when t is large, it may provide an estimator of the MRL function, whereas the empirical estimator is defined to be 0. The need for estimating the MRL function when the sample size is small is presented in Park (1990).

It is also emphasized that although we discuss only the gamma and Weibull families for applying the MLE of MRL function in this paper, the same method can be used to estimate the MRL function of other parametric families as long as its maximum likelihood estimators of the mean and variance exist.

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모수족에서 평균 잔여수명의 추정량[4)]

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요약

본 논문에서는 평균 잔여수명의 추정에 있어서 Weibull과 gamma분포의 평균 잔여수명을 구하는데 적분이 쉽게 되지 않으므로 부분적률에 근거한 새로운 추정량을 제시하였으며, 비록 니 추정량은 일치추정량이 아니지만 소표본인 경우에서 일치추정량인 기존의 경험적 추정량 보다 평균제곱오차가 작다는 것을 몬테 칼로 기법을 써서 보였다.

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