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A Heuristic Approach for Approximating the ARL of the CUSUM Chart

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ABSTRACT

A new method for approximating the average run length (ARL) of cumulative sum ($CUSUM$) chart is proposed. This method uses the conditional expectation for the test statistic before the stopping time and its asymptotic conditional density function. The values obtained by this method are compared with some other methods in normal and exponential case.

KEYWORDS: Cumulative sum control chart, Average sample number, Sequential probability ratio test, Average sample number, Operating characteristic function.

1. INTRODUCTION

Many statistical control charts have been developed to control the quality of the products. Among them, the cumulative sum ($CUSUM$) control chart was

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proposed by Page (1954). The $CUSUM$ chart has been known to be efficient in detecting small and consistent changes of the parameter when compared with Shewhart chart (1931).

Suppose that $X_1, X_2, \cdots$ are sequentially observed i.i.d. random variables with density $f(x;\theta)$ where θ denotes the quality of the process. The process $\{X_i, i=1,2,\cdots\}$is said to be in-control if $\theta = \theta_0$ and out-of-control if $\theta = \theta_1 (> \theta_0)$. For convenience, only positive shifts of the parameter θ are considered.

The $CUSUM$ procedure based on the log probability ratio statistic ($LPRS$) is defined as follows: let

$$W_n = \sum_{i=1}^{n} Z_i - \min_{0 \le l \le n} \sum_{i=1}^{l} Z_i,$$

where $Z_i = \log f(X_i;\theta_1)/f(X_i;\theta_0)$, and define the run length as

$$T = \min\{n; W_n \geq h\},$$

where h is a suitably chosen constant and $\sum_{i=1}^{0} Z_i = 0$. We assume that $Var(Z_i)$ exists and greater than 0. It has been shown by Moustakides (1986) that the $CUSUM$ procedure based on $LPRS$ is optimal in detecting a change in distribution in the sense that it minimizes $E_{\theta_1}T$ for any fixed $E_{\theta_0}T$.

Page (1954) showed that the $CUSUM$ procedure can be expressed as a sequence of Wald's (1947) sequential probability ratio tests ($SPRT$) with lower boundary zero, upper boundary h, and an initial value of zero. The $CUSUM$ procedure is mathematically equivalent to performing the $SPRT$'s successively until the upper boundary is reached and if the lower boundary is reached, a new $SPRT$ is performed. According to the mathematical equivalence, the average run length (ARL) of the $CUSUM$ chart can be expressed as

$$ARL = \frac{ASN}{1 - OC(\theta)}, \qquad (1.1)$$

where ASN and $OC(\theta)$ denote the average sample number and the operating characteristic functions of the $SPRT$ with boundaries $(0, h)$, respectively.

The ASN and OC functions of the $SPRT$ are not known explicitly in general, and thus neither the ARL. Hence many methods have been developed for approximating the ARL such as Van Dobben de Bruyn (1968), Goel and Wu (1971), Reynolds (1975), Kahn (1978), Siegmund (1979), Park (1987), and Park and Kim (1990). A new approach to approximate the ARL is introduced here and compared with the results of existing methods in normal and exponential case.

2. THE METHOD OF APPROXIMATION

Define the sample number of the $SPRT$

$$N = \min\{n : S_n \leq 0 \text{ or } S_n \geq h\},$$

where $S_n = \sum_{i=1}^{n} Z_i$, and $OC(\theta) = P(S_N \leq 0)$. Then $E(N)$ and $OC(\theta)$ denote the ASN and OC functions of the $SPRT$ with boundaries $(0, h)$, respectively.

In this section, a new approximation technique for the calculation of the ARL is presented. This method modifies the CBST(Conditon of Before Stopping Time) method (Park and Kim, 1990) by approximating the conditional density function of S_{N-1}. After taking the conditional expectation for S_{N-1}, the CBST method replaces S_{N-1} by the conditional expectation of S_{N-1}, but this method calculates the conditional expectation by using the asymptotic conditional density function of S_{N-1}.

The expectation of S_N can be expressed as

$$E(S_N) = E(S_N|S_N \geq h)(1 - OC(\theta)) + E(S_N|S_N \leq 0)OC(\theta) \quad (2.1)$$

and also by Wald equation,

$$E(S_N) = E(N) \cdot E(Z). \quad (2.2)$$

From (2.1) and (2.2), we have the expression for the ASN function

$$E(N) = \frac{E(S_N|S_N \geq h)(1 - OC(\theta)) + E(S_N|S_N \leq 0)OC(\theta)}{E(Z)}, \quad (2.3)$$

if $E(Z) \neq 0$.

For the case $E(Z) = 0$, we use $E(S_N^2)$ instead of $E(S_N)$.

$$E(S_N^2) = E(S_N^2|S_N \geq h)(1 - OC(\theta)) + E(S_N^2|S_N \leq 0)OC(\theta). \quad (2.4)$$

Also, by Wald equation

$$E(S_N^2) = E(N) \cdot E(Z^2). \tag{2.5}$$

From (2.4) and (2.5), we have

$$E(N) = \frac{E(S_N^2|S_N \geq h)(1 - OC(\theta)) + E(S_N^2|S_N \leq 0)OC(\theta)}{E(Z^2)}, \tag{2.6}$$

if $E(Z) = 0$.

To calculate $E(S_N|S_N \geq h)$ and $E(S_N|S_N \leq 0)$, we take the conditional expectation for S_{N-1}. That is,

$$\begin{aligned} E(S_N|S_N \geq h) &= E\{E(S_N|S_N \geq h, S_{N-1})\} \\ &= \int_0^h E(Z_N + S_{N-1}|S_N \geq h, S_{N-1} = y) \cdot f_{S_{N-1}}(y|S_N \geq h)dy \\ &= \int_0^h \{y + E(Z_N|Z_N \geq h - y)\} \cdot f_{S_{N-1}}(y|S_N \geq h)dy, \end{aligned}$$

where $f_{S_{N-1}}(y|\cdot)$ is the conditional density function of S_{N-1}.
Here we use the following approximation. For $0 < y < h$,

$$\begin{aligned} f_{S_{N-1}}(y|S_N \geq h) &\approx f_{S_{n-1}}(y|S_n \geq h, 0 < S_{n-1} < h) \\ &= \frac{P(Z_n \geq h - S_{n-1}|S_{n-1} = y) \cdot f_{S_{n-1}}(y)}{\int_0^h P(Z_n \geq h - S_{n-1}|S_{n-1} = x) \cdot f_{S_{n-1}}(x)dx} \\ &\approx \frac{P(Z_n \geq h - y)}{\int_0^h P(Z_n \geq h - x)dx}, \end{aligned}$$

since for large n and $0 < x < h$, $f_{S_{n-1}}(y) \approx f_{S_{n-1}}(x)$ if $Var(Z_i) > 0$.
Therefore, we obtain

$$E(S_N|S_N \geq h) \approx \int_0^h \frac{\{y + E(Z_N|Z_N \geq h - y)\} \cdot P(Z_n \geq h - y)}{\int_0^h P(Z_n \geq h - x)dx} dy. \tag{2.7}$$

Similarly,

$$\begin{aligned} E(S_N|S_N \leq 0) &= E\{E(S_N|S_N \leq 0, S_{N-1})\} \\ &= \int_0^h E(Z_N + S_{N-1}|S_N \leq 0, S_{N-1} = y) \cdot f_{S_{N-1}}(y|S_N \leq 0)dy \\ &= \int_0^h \{y + E(Z_N|Z_N \leq -y)\} \cdot f_{S_{N-1}}(y|S_N \leq 0)dy, \end{aligned}$$

and

$$\begin{aligned} f_{S_{N-1}}(y|S_N \leq 0) &\approx f_{S_{n-1}}(y|S_n \leq 0, 0 < S_{n-1} < h) \\ &\approx \frac{P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx}. \end{aligned}$$

Therefore,

$$E(S_N|S_N \leq 0) \approx \int_0^h \frac{\{y + E(Z_N|Z_N \leq -y)\} \cdot P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx} dy. \tag{2.8}$$

Next, for the case $E(Z) = 0$, we approximate the $E(S_N^2|S_N \geq h)$ and $E(S_N^2|S_N \leq 0)$ with the same technique.

$$\begin{aligned} &E(S_N^2|S_N \geq h) \\ &= E\{E(S_N^2|S_N \geq h, S_{N-1})\} \\ &= \int_0^h E((Z_N + S_{N-1})^2|S_N \geq h, S_{N-1} = y) \cdot f_{S_{N-1}}(y|S_N \geq h)dy \\ &= \int_0^h \{y^2 + 2yE(Z_N|Z_N \geq h-y) + E(Z_N^2|Z_N \geq h-y)\} \cdot f_{S_{N-1}}(y|S_N \geq h)dy \\ &\approx \int_0^h \frac{\{y^2 + 2yE(Z_N|Z_N \geq h-y) + E(Z_N^2|Z_N \geq h-y)\} \cdot P(Z_n \geq h-y)}{\int_0^h P(Z_n \geq h-x)dx} dy. \end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned} &E(S_N^2|S_N \leq 0) \\ &\approx \int_0^h \frac{\{y^2 + 2yE(Z_N|Z_N \leq -y) + E(Z_N^2|Z_N \leq -y)\} \cdot P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx} dy. \end{aligned} \tag{2.10}$$

From Wald's fundamental identity,

$$\begin{aligned} 1 &= E(e^{d(\theta)S_N}) \\ &= E(e^{d(\theta)S_N}|S_N \geq h)(1 - OC(\theta)) + E(e^{d(\theta)S_N}|S_N \leq 0)OC(\theta), \end{aligned}$$

where $d(\theta)$ is the unique nonzero solution d of $E(e^{dZ}) = 1$. Thus

$$OC(\theta) = \frac{E(e^{d(\theta)S_N}|S_N \geq h) - 1}{E(e^{d(\theta)S_N}|S_N \geq h) - E(e^{d(\theta)S_N}|S_N \leq 0)}, \tag{2.11}$$

if $E(Z) \neq 0$ (i.e. $d(\theta) \neq 0$).

For approximation of the OC function, we use the same technique.

$$
\begin{aligned}
E(e^{d(\theta)S_N}|S_N \geq h) \\
&= E\{E(e^{d(\theta)S_N}|S_N \geq h, S_{N-1})\} \\
&= \int_0^h E(e^{d(\theta)(Z_N+S_{N-1})}|S_N \geq h, S_{N-1} = y) \cdot f_{S_{N-1}}(y|S_N \geq h)dy \\
&= \int_0^h e^{d(\theta)y} \cdot E(e^{d(\theta)Z_N}|Z_N \geq h-y) \cdot f_{S_{N-1}}(y|S_N \geq h)dy \\
&\approx \int_0^h \frac{e^{d(\theta)y} \cdot E(e^{d(\theta)Z_N}|Z_N \geq h-y) \cdot P(Z_n \geq h-y)}{\int_0^h P(Z_n \geq h-x)dx} dy. \qquad (2.12)
\end{aligned}
$$

Similarly,

$$
E(e^{d(\theta)S_N}|S_N \leq 0) \approx \int_0^h \frac{e^{d(\theta)y} \cdot E(e^{d(\theta)Z_N}|Z_N \leq -y) \cdot P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx} dy. \qquad (2.13)
$$

If $E(Z) = 0$, we use L'Hospital's rule to (2.11) and obtain the OC function as

$$
OC(\theta) = \frac{E(S_N|S_N \geq h)}{E(S_N|S_N \geq h) - E(S_N|S_N \leq 0)}. \qquad (2.14)
$$

Therefore the ASN function is calculated by submitting (2.7), (2.8), $OC(\theta)$ to (2.3), where $OC(\theta)$ is calculated by submitting (2.12), (2.13) to (2.11). For the case $E(Z) = 0$, the ASN function is calculated by (2.9), (2.10), $OC(\theta)$ to (2.6), where $OC(\theta)$ is calculated by (2.14). Finally the ARL is calculated by (1.1). Above (2.7), (2.8), (2.9), (2.10), (2.12) and (2.13) can be easily obtained by using the computer programs such as IMSL library FORTRAN subroutines.

3. NORMAL CASE

In evaluating the accuracy of the ARL obtained by the new method, we will compare the new method with the results of the SLAE(Systems of Linear Algebraic Equations) method (Goel and Wu, 1971) and the CBST method (Park and Kim, 1990) for cases where the underlying destribution is normal. This is because the SLAE method is a standard method which can produce almost exact values numerically, and the CBST method is better than or at least as good as the other approximation methods in normal case.

Consider that $\{X_i, i = 1, 2, \cdots\}$ are i.i.d. random variables from a normal distribution with mean θ and unit variance and consider the detection problem for $\theta = \theta_0 = 0$ verse $\theta = \theta_1 (> 0)$. Then

$$Z_i = (\theta_1 - \theta_0)(X_i - \frac{\theta_1 + \theta_0}{2}),$$

and Z_i has a normal distribution with mean $\mu = (\theta_1 - \theta_0)(\theta - (\theta_1 + \theta_0)/2)$ and variance $\sigma^2 = (\theta_1 - \theta_0)^2$.

The expressions in (2.7) and (2.8) are derived as

$$\begin{aligned} E(S_N | S_N \geq h) &\approx \int_0^h \frac{\{y + E(Z_N | Z_N \geq h - y)\} \cdot P(Z_n \geq h - y)}{\int_0^h P(Z_n \geq h - x) dx} dy \\ &= \int_0^h \frac{\{y + \mu + \sigma \frac{\phi(\frac{y+\mu-h}{\sigma})}{\Phi(\frac{y+\mu-h}{\sigma})}\} \Phi(\frac{y+\mu-h}{\sigma})}{\int_0^h \Phi(\frac{x+\mu-h}{\sigma}) dx} dy \\ &= \int_0^h \frac{(y+\mu)\Phi(\frac{y+\mu-h}{\sigma}) + \sigma\phi(\frac{y+\mu-h}{\sigma})}{\int_0^h \Phi(\frac{x+\mu-h}{\sigma}) dx} dy, \end{aligned}$$

and

$$\begin{aligned} E(S_N | S_N \leq 0) &\approx \int_0^h \frac{\{y + E(Z_N | Z_N \leq -y)\} \cdot P(Z_n \leq -y)}{int_0^h P(Z_n \leq -x) dx} dy \\ &= \int_0^h \frac{\{y + \mu - \sigma \frac{\phi(\frac{-y-\mu}{\sigma})}{\Phi(\frac{-y-\mu}{\sigma})}\} \Phi(\frac{-y-\mu}{\sigma})}{\int_0^h \Phi(\frac{-x-\mu}{\sigma}) dx} dy \\ &= \int_0^h \frac{(y+\mu)\Phi(\frac{-y-\mu}{\sigma}) - \sigma\phi(\frac{-y-\mu}{\sigma})}{\int_0^h \Phi(\frac{-x-\mu}{\sigma}) dx} dy, \end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the density and distribution function of a standard normal distribution, respectively.

For the case $\mu = 0$, the expressions in (2.9) and (2.10) are obtained as follows.

$$\begin{aligned} &E(S_N^2 | S_N \geq h) \\ &\approx \int_0^h \frac{\{y^2 + 2yE(Z_N | Z_N \geq h - y) + E(Z_N^2 | Z_N \geq h - y)\} \cdot P(Z_n \geq h - y)}{\int_0^h P(Z_n \geq h - x) dx} dy \\ &= \int_0^h \frac{(y^2 + \sigma^2)\Phi(\frac{y-h}{\sigma}) + (y+h)\sigma\phi(\frac{y-h}{\sigma})}{\int_0^h \Phi(\frac{x-h}{\sigma}) dx} dy, \end{aligned}$$

$$E(S_N^2|S_N \le 0)$$
$$\approx \int_0^h \frac{\{y^2 + 2yE(Z_N|Z_N \le -y) + E(Z_N^2|Z_N \le -y)\} \cdot P(Z_n \le -y)}{\int_0^h P(Z_n \le -x)dx} dy$$
$$= \int_0^h \frac{(y^2+\sigma^2)\Phi(\frac{-y}{\sigma}) - y\sigma\phi(\frac{-y}{\sigma})}{\int_0^h \Phi(\frac{-x}{\sigma})dx} dy.$$

Similarly, the expressions in (2.12) and (2.13) are obtained as

$$E(e^{d(\theta)S_N}|S_N \ge h) \approx \int_0^h \frac{e^{d(\theta)y} E(e^{d(\theta)Z_N}|Z_N \ge h-y) \cdot P(Z_n \ge h-y)}{\int_0^h P(Z_n \ge h-x)dx} dy$$
$$= e^{\{d(\theta)\mu + \frac{(d(\theta)\sigma)^2}{2}\}} \int_0^h \frac{e^{d(\theta)y}\Phi(\frac{y+\mu-h}{\sigma} + d(\theta)\sigma)}{\int_0^h \Phi(\frac{x+\mu-h}{\sigma})dx} dy,$$

$$E(e^{d(\theta)S_N}|S_N \le 0) \approx \int_0^h \frac{e^{d(\theta)y} E(e^{d(\theta)Z_N}|Z_N \le -y) \cdot P(Z_n \le -y)}{\int_0^h P(Z_n \le -x)dx} dy$$
$$= e^{\{d(\theta)\mu + \frac{(d(\theta)\sigma)^2}{2}\}} \int_0^h \frac{e^{d(\theta)y}\Phi(\frac{-y-\mu}{\sigma} - d(\theta)\sigma)}{\int_0^h \Phi(\frac{-x-\mu}{\sigma})dx} dy,$$

where $d(\theta) = (\theta_1 + \theta_0 - 2\theta)/(\theta_1 - \theta_0)$.

The SLAE method (Goel and Wu, 1971) for estimating the ARL is as follows. Let $p(z)$ and $N(z)$ be the OC and ASN functions with starting point at z and boundaries $(0, h)$, then we have the following expressions, which belong to the family of Fredholm equations of the second kind.

$$p(z) = \int_{-\infty}^{-z} \frac{1}{\sigma} \cdot \phi(\frac{x-\mu}{\sigma})dx + \int_0^h p(x) \cdot \frac{1}{\sigma} \cdot \phi(\frac{x-z-\mu}{\sigma})dx, \quad (3.1)$$

$$N(z) = 1 + \int_0^h N(x) \cdot \frac{1}{\sigma} \cdot \phi(\frac{x-z-\mu}{\sigma})dx. \quad (3.2)$$

By the Gaussian quadrature formula, equations (3.1) and (3.2) can be reduced to

$$p(z) \approx \Phi(\frac{b-z-\mu}{\sigma}) + \sum_{k=1}^m w_k \cdot \frac{1}{\sigma} \cdot \phi(\frac{x_k-z-\mu}{\sigma}) \cdot p(x_k), \quad (3.3)$$

$$N(z) \approx 1 + \sum_{k=1}^m w_k \cdot \frac{1}{\sigma} \cdot \phi(\frac{x_k-z-\mu}{\sigma}) \cdot N(x_k), \quad (3.4)$$

where w_k and x_k are the weights and roots of the Gaussian quadrature for the interval $(0, h)$ respectively, and m is the number of Gaussian points used.

The values $p(x_k)$ in equation (3.3) are obtained by solving the following SLAE

$$(\mathbf{I} - \mathbf{C}) \cdot \mathbf{P} = \mathbf{B},$$

where $\mathbf{P}' = \{p(x_1), \cdots, p(x_m)\}$, $\mathbf{B}' = \{\Phi((b - x_1 - \mu)/\sigma), \cdots, \Phi((b - x_m - \mu)/\sigma)\}$, $\mathbf{C} = \{c(i,j)\}$ is an $m \times m$ matrix for $c(i,j) = w_j \cdot \phi((x_j - x_i - \mu)/\sigma)/\sigma$, for $i, j = 1, 2, \cdots, m$, and $\mathbf{I}$ is an $m \times m$ identity matrix.

Similarly, the values $N(x_k)$ in equation (3.4) are obtained from

$$(\mathbf{I} - \mathbf{C}) \cdot \mathbf{N} = \mathbf{1} ,$$

where $\mathbf{N}' = \{N(x_1), \cdots, N(x_m)\}$ and $\mathbf{1}$ is an $m \times 1$ unit vector.

For convenience of numerical comparisons, we let $\theta_1 = -\theta_0$, $\theta' = \theta/(\theta_1 - \theta_0)$, $h' = h/(\theta_1 - \theta_0)$. Several combinations of h' and θ' are employed for this purpose. From TABLE 1, it is seen that the proposed method is better than the CBST method for almost all the cases in estimating the ARL. Therefore the proposed approximate method for the ARL appears to be good enough to apply in practice. The reason why the difference in the ARL value is relatively high for large negative θ' is that the OC value in the denominator of (1.1) is nearly 1. Thus the ARL value is very sensitive to the OC and the ASN value.

4. EXPONENTIAL CASE

Suppose that $\{X_i, i = 1, 2, \ldots\}$ are i.i.d. with density $f(x; \lambda) = \lambda \cdot e^{-\lambda x}, x > 0$ and consider the detection problem for $\lambda = \lambda_0 = 1$ versus $\lambda = \lambda_1 (> 1)$. In this case, $Z_i = -(\lambda_1 - 1)X_i + \log \lambda_1$, and $d(\lambda)$ is the unique nonzero solution of $\frac{\lambda \cdot \lambda_1^d}{(\lambda_1 - 1)d + \lambda} = 1$. Then the probability density function of Z_i is

$$g(z; \lambda) = \frac{\lambda}{\lambda_1 - 1} e^{-\frac{\lambda(\log \lambda_1 - z)}{\lambda_1 - 1}}, \quad z < \log \lambda_1.$$

The expressions in (2.7) and (2.8) are obtained as follows

$$
\begin{aligned}
&E(S_N|S_N \geq h)\\
&\approx \int_0^h \frac{\{y + E(Z_N|Z_N \geq h-y)\} \cdot P(Z_n \geq h-y)}{\int_0^h P(Z_n \geq h-x)dx} dy\\
&= \frac{\frac{1}{2}\log\lambda_1\{\log\lambda_1 + 2h - \frac{2(\lambda_1-1)}{\lambda}\} + (h - \frac{\lambda_1-1}{\lambda})\frac{\lambda_1-1}{\lambda}\{e^{-\frac{\lambda\log\lambda_1}{\lambda_1-1}} - 1\}}{\log\lambda_1 + \frac{\lambda_1-1}{\lambda}\{e^{-\frac{\lambda\log\lambda_1}{\lambda_1-1}} - 1\}}, \qquad (4.1)
\end{aligned}
$$

$$
\begin{aligned}
E(S_N|S_N \leq 0) &\approx \int_0^h \frac{\{y + E(Z_N|Z_N \leq -y)\} \cdot P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx} dy\\
&= -\frac{\lambda_1 - 1}{\lambda}. \qquad (4.2)
\end{aligned}
$$

Similarly, the expressions in (2.12) and (2.13) are derived as

$$
\begin{aligned}
&E(e^{d(\theta)S_N}|S_N \geq h)\\
&\approx \int_0^h \frac{e^{d(\theta)y} \cdot E(e^{d(\theta)Z_N}|Z_N \geq h-y) \cdot P(Z_n \geq h-y)}{\int_0^h P(Z_n \geq h-x)dx} dy\\
&= \frac{\frac{\lambda}{(\lambda_1-1)d(\lambda)+\lambda}\left[\frac{1}{d(\lambda)}e^{d(\lambda)h}\{e^{d(\lambda)\log\lambda_1} - 1\} + \frac{\lambda_1-1}{\lambda}e^{d(\lambda)h}\{e^{-\frac{\lambda\log\lambda_1}{\lambda_1-1}} - 1\}\right]}{\log\lambda_1 + \frac{\lambda_1-1}{\lambda}\{e^{-\frac{\lambda\log\lambda_1}{\lambda_1-1}} - 1\}}, \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
E(e^{d(\theta)S_N}|S_N \leq 0) &\approx \int_0^h \frac{e^{d(\theta)y}E(e^{d(\theta)Z_N}|Z_N \leq -y) \cdot P(Z_n \leq -y)}{\int_0^h P(Z_n \leq -x)dx} dy\\
&= \frac{\lambda}{(\lambda_1 - 1)d(\lambda) + \lambda}. \qquad (4.4)
\end{aligned}
$$

From (4.1), (4.2), (4.3), and (4.4) we obtain the ASN and OC functions, and then the ARL by (1.1). In exponential case, we regard the values of ARL obtained by Stadje(1987)'s method as a standard. This is because in exponential case the SLAE method is not accurate on account of the discontinuity of the kernel. However Stadje's expressions are too complicated to use in practice. In TABLE 2, the values of ARL by Stadje's, the CBST, and the proposed methods are obtained for some given boundaries($h' = h(\lambda_1 - 1)$) and parameter value.

Generally the proposed method is not as accurate as the $CBST$ in exponential case. But, it may be noted that the proposed method will be helpful in estimating the ARL, since this method gives the explicit form of the ARL and still gives good approximations.

5. CONCLUSIONS

In this paper, we consider the approximation for the ARL of the $CUSUM$ chart. Many techniques have been proposed for estimating the ARL because their exact evaluations have been hopeless in general.

The two main approaches for estimating the ARL are numerical and approximation methods, which have their own advantages and disadvantages. Numerical methods give accurate results such as Goel and Wu (1971), whereas approximation methods are useful in evaluating the properties of the $CUSUM$ procedures and use less computer memory space in general. This is because analytic expressions for characteristics of the procedure are available.

In this paper, the new method is proposed by using the conditional expectation of S_N given S_{N-1}. One important fact in this method is to obtain the asymptotic conditional density function of S_{N-1}. Applying the new method to normal and exponential case, it seems to be successful in estimating the ARL.

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Table 1. Values of the ARL in normal case

	θ′	SLAE	CBST	New
	-1.00	1962.87	1975.17	2142.81
	-0.75	442.80	430.06	456.84
	-0.50	117.60	113.38	118.17
	-0.25	39.47	38.32	39.35
h′= 3	0.00	17.35	16.99	17.29
	0.25	9.68	9.53	9.65
	0.50	6.40	6.32	6.39
	0.75	4.73	4.67	4.72
	1.00	3.75	3.71	3.75
	-0.75	9010.84	8744.65	9289.01
	-0.50	930.93	898.76	934.97
	-0.25	141.69	138.23	141.30
	0.00	38.01	37.48	37.92
h′= 5	0.25	17.05	16.89	17.02
	0.50	10.38	10.29	10.36
	0.75	7.39	7.34	7.39
	1.00	5.75	5.70	5.74
	1.50	4.01	3.97	4.01
	-0.50	18983.32	18318.55	19047.46
	-0.25	736.82	720.66	734.97
	0.00	84.00	83.22	83.87
	0.25	28.76	28.60	28.74
h′= 8	0.50	16.37	16.29	16.36
	0.75	11.39	11.34	11.39
	1.00	8.75	8.70	8.74
	1.50	6.01	5.97	6.01
	2.00	4.62	4.58	4.62
	-0.25	2071.87	2027.43	2066.60
	0.00	124.66	123.71	124.50
	0.25	36.71	36.55	36.68
h′=10	0.50	20.37	20.29	20.36
	0.75	14.06	14.00	14.05
	1.00	10.75	10.70	10.74
	1.50	7.34	7.31	7.34
	2.00	5.62	5.58	5.62

Table 2. Values of the ARL in exponential case

	λ	Stadje	CBST	New
$\lambda_1 = 1.4$	1	422.09	426.19	427.30
	1.1	179.58	180.69	180.94
	1.2	98.06	98.46	98.54
$h' = 7.48925$	1.3	64.38	64.58	64.61
	1.4	47.85	47.96	47.98
$\lambda_1 = 1.6$	1	676.02	685.04	687.84
	1.3	83.28	83.63	83.70
	1.4	58.00	58.18	58.21
$h' = 6.52$	1.5	44.48	44.59	44.61
	1.6	36.42	36.50	36.52
$\lambda_1 = 1.9$	1	342.15	348.65	351.06
	1.5	38.08	38.20	38.24
	1.6	30.86	30.93	30.95
$h' = 4.09867$	1.7	26.06	26.09	26.11
	1.8	22.70	22.71	22.73
	1.9	20.26	20.25	20.27