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Sub–gaussian Techniques in Obtaining Laws of Large Numbers in $L^1(R)$†

Sungho Lee[1] and Robert Lee Taylor[2]

ABSTRACT

Some exponential moment inequalities for sub–gaussian random variables are studied in this paper. These inequalities are used to obtain laws of large numbers for random variables and random elements in $L^1(R)$.

KEYWORDS: Sub-gaussian random variables, Complete consistency, L^1 random elements

1. INTRODUCTION

Random variables that are in a certain sense subordinate to Gaussian random variables are called sub–gaussian random variables. Recently, the properties of sub–gaussian random variables have drawn the attention of experts in quantum field theory owing to the possibility of extending the properties of Gaussian random variables and processes to the larger classes of random variables and processes. Several authors, most notably Buldygin and

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[1]Department of Statistics, Taegu University, Kyungbuk 713–714, Korea.
[2]Department of Statistics, University of Georgia, Athens, Georgia 30602, U.S.A.
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Kozachenko(1980, 1988) and Kozachenko and Ostrovski(1986), studied sub-gaussian random variables and formulated some general assertions about the sample continuity of sub-gaussian processes.

On the other hand, those properties of sub-gaussian random variables have been used in establishing strong laws of large numbers. Chow(1966) proved strong laws of large numbers for independent sub-gaussian random variables. Taylor and Hu(1987a) introduced sub-gaussian techniques in proving strong laws of large numbers. Furthermore, Taylor and Hu(1987b) developed strong laws of large numbers for random elements in $C_0(R)$ which have direct applications in establishing the uniform strong consistency of the kernel density estimators. In this paper we shall study some inequalities for sub-gaussian random variables. These inequalities will be used to obtain strong laws of large numbers for random variables and random elements in $L^1(R)$.

2. PRELIMINARIES

A random variable X is said to be sub-gaussian(with parameter α) if for some $\alpha > 0$

$$E(\exp(\lambda X)) \leq \exp\left(\frac{\lambda^2\alpha^2}{2}\right) \qquad \text{for all } \lambda \in R. \tag{2.1}$$

The parameter α is not unique. The exact lower bound

$$\tau(X) = \inf\{\alpha > 0 : E(\exp(\lambda X)) \leq \exp(\frac{\lambda^2\alpha^2}{2})\}$$

is called the sub-gaussian norm of X and denoted by $\|X\|$. The norm $\|X\|$ may be defined by the equation

$$\|X\| = \sup_{\lambda\neq 0}\left(\frac{2\log E(\exp(\lambda X))}{\lambda^2}\right)^{1/2}. \tag{2.2}$$

Note that all moments of a sub-gaussian random variable will exist and the mean must be zero. It was proved that a class of sub-gaussian random variables defined on a probability space (Ω, A, P) with the norm $\tau(X)$ is a Banach space (cf. Buldygin and Kozachenko(1980)). Let sub(R) denote the class of sub-gaussian random variables. A norm equivalent to $\|X\|$ can also be introduced in terms of all the moments of X. Thus for X to be sub-gaussian, it is necessary and sufficient that $\theta_1(X) < \infty$ or $\theta_2(X) < \infty$ where $\theta_1(X) = \sup_{k\geq 1} \left(2^k k! E(X^{2k})/(2k)!\right)^{1/2k}$ and $\theta_2(X) = \sup_{k\geq 1} k^{-1/2} \left(E(X^{2k})\right)^{1/2k}$. Moreover $\left(\sqrt{2}e^{1/16}\right)^{-1} \theta_1(X) \leq \frac{\sqrt{e}}{2}\theta_2(X) \leq \tau(X) \leq (3.1)^{1/4}\theta_1(X) \leq 2^{-1/2}(3.1)^{1/4}e^{9/16}$ $\theta_2(X)$. The simplest example which explains the origin of the term "sub-gaussian" is provided by Gaussian random variables with mean zero; for them $\|X\| = \sqrt{\sigma^2}$, the standard deviation of X. The following basic properties of sub-gaussian random variables are adapted from Taylor and Hu(1987 a) and Ostrovskii(1990):

(1) If X is sub-gaussian, then for $x > 0$

$$P(X > x) \leq \exp(-\frac{x^2}{2\tau^2(X)}).$$

(2) If $X_1, X_2, \cdots, X_n$ are independent, sub-gaussian random variables with parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively, then

$$S_n = \sum_{i=1}^{n} X_i \text{ is sub-gaussian with parameter } \quad \alpha = (\sum_{i=1}^{n} \alpha_i^2)^{1/2}.$$

(3) If X is bounded ($|X| \leq m$) and $E(X) = 0$, then X is sub-gaussian with the norm $\tau(X) \leq m$.

A random vector $X' = (X_1, \cdots, X_n)$ is said to be sub-gaussian if there is a symmetric nonnegative definite matrix $B : R^n \to R^n$ such that for all $\lambda \in R^n$, $E(\exp(\lambda' X)) \leq \exp(\lambda' B \lambda/2)$. Any linear functional of a sub-gaussian random vector is a sub-gaussian random variable (cf.Buldygin and Kozachenko(1988)) and hence every sub-gaussian random vector has zero mean. Let X be a random vector and let $X_1, X_2, \cdots, X_n$ be the coordinates of X in the orthonormal basis $e_1, e_2, \cdots, e_n$. Then a sufficient condition for X to be sub-gaussian is $X_i \in$ sub(R), $i = 1, 2, \cdots, n$. Let sub(R^n) denote the class of sub-gaussian random vectors on R^n. The following lemmas are sub(R^n) counterparts to the basic properties on sub(R).

Lemma 2.1. (Buldygin and Kozachenko(1988))
Let $X \in$ sub(R^n) with a symmetric nonnegative definite matrix B and $B^{-1}x \geq 0$, where $x \in R^n$, $|B| \neq 0$. Then, $P(X_1 \geq x_1, \cdots, X_n \geq x_n) \leq \exp(-x'B^{-1}x/2)$.

Lemma 2.2. If $X_1, X_2, \cdots, X_n$ are independent, sub–gaussian random vectors with symmetric nonnegative definite matrices $B_1, B_2, \cdots, B_n$, respectively, then $S_n = \sum_{i=1}^n X_i$ is sub–gaussian with $B = \sum_{i=1}^n B_i$.

Proof. For all $\lambda \in R^n$,

$$\begin{aligned} E(\exp(\lambda' S_n)) &= \prod_{i=1}^n E(\exp(\lambda' X_i)) \\ &\leq \prod_{i=1}^n \exp(\frac{\lambda' B_i \lambda}{2}) \\ &= \exp(\frac{\lambda' B \lambda}{2}) \end{aligned}$$

The proof of Lemma 2.3 is based on Ostrovskii(1990):

Lemma 2.3. Let B be a symmetric positive definite matrix. If $X' = (X_1, \cdots, X_n)$ is a random vector with $E(X) = 0$ and $X'B^{-1}X \leq m$ a.s., then X is sub–gaussian with the symmetric positive definite matrix mB.

Proof. To show that for all $\lambda \in R^n$

$$E(\exp(\lambda' X)) \leq \exp(\frac{\lambda' m B \lambda}{2}),$$

we shall apply the property(3) on sub(R). For each $\lambda \in R^n$,

$$E(\lambda' X) = E(\sum_{i=1}^n \lambda_i X_i) = \sum_{i=1}^n \lambda_i E(X_i) = 0.$$

By the extended Cauchy–Schwarz inequality,

$$\max_{x \neq 0} \left\{ \frac{(\lambda' x)^2}{x' B^{-1} x} \right\} = \lambda' B \lambda$$

and hence for each $\lambda \in R^n$, $|\lambda' x| \leq m^{1/2}\ (\lambda' B \lambda)^{1/2}$.

Thus, $E(\exp((\lambda' X)t)) \leq \exp(mt^2 \lambda' B \lambda/2)$ for all $t \in R$. Put $t = 1$. Then the result follows.

In the class sub(R), we have $E(X^2) \leq \tau^2(X)$. When $E(X^2) = \tau^2(X)$, we call it strictly sub-gaussian. That is, for all $\lambda \in R$, $E(\exp(\lambda X)) \leq \exp(\sigma^2\lambda^2/2)$ where $\sigma^2 = var(X)$. A random vector $X \in R^n$ is said to be strictly sub-gaussian if $E(\exp(\lambda' X)) \leq \exp(\lambda' B \lambda/2)$ for all $\lambda \in R^n$ where B is the co-variance matrix of X. The class of strictly sub-gaussian random variables is not closed with respect to addition, but the sum of independent strictly sub-gaussian random variables is strictly sub-gaussian. We denote the class of strictly sub-gaussian random vectors on R^n by $\overline{\text{sub}}(R^n)$. One of the simplest examples except Gaussian random variables with zero mean is the random variable X taking values ± 1 with probability 1/2. Finally, we will introduce sub-gaussian random elements in $L^1(R)$ in this paper. Let $(R, \mathcal{B}(R), \mu)$ be a measure space. The space $L^1(R)$ is the set of all μ-equivalence classes of $\mathcal{B}(\mathcal{R})$-measurable functions $x : R \to R$ such that $\|x\| = \int |x| d\mu$ where μ denotes the Lebesgue measure on R. Let (Ω, A, P) be a probability space and let $\tilde{X}$ be a function from Ω into $L^1(R)$. If $\tilde{X}^{-1}(B) \in A$ for every Borel set $B \in \mathcal{B}(L^1(R))$, then $\tilde{X}$ is said to be a random element in $L^1(R)$. If $\tilde{X}$ is a random element in $L^1(R)$, then there exists a function $X : R \times \Omega \to R$ such that (i) $\forall \omega \in \Omega$, $X(\cdot, \omega)$ is a Lebesgue integrable function, (ii) $\forall t \in R$, $X(t, \cdot)$ is an extended random variable (cf.Taylor and Lee(1990)). Based on this fact, a random element $\tilde{X}$ in $L^1(R)$ is said to be sub-gaussian if for some function $\alpha(s) > 0$, $E(\exp(\lambda X(s))) \leq \exp(\lambda^2 \alpha^2(s)/2)$ for all $s, \lambda \in R$.

3. INEQUALITIES FOR SUB-GAUSSIAN RANDOM VARIABLES

Some exponential moment inequalities for sub-gaussian random variables are obtained in this section. These inequalities will be used to obtain complete convergence for laws of large numbers for random variables and random elements in $L^1(R)$.

Lemma 3.1. Let $X_i \in \text{sub}(R)$ and $a_i \in R$, $i = 1, 2, \cdots, n$. Then for $\lambda > 0$

$$
\begin{aligned}
P(\sum_{k=1}^{n} a_k X_k > \lambda) &\le \exp\left(-\frac{\lambda^2}{2\tau^2(\sum_{k=1}^n a_k X_k)}\right) \\
&\le \exp\left(-\frac{\lambda^2}{2n\sum_{k=1}^n a_k^2\tau^2(X_k)}\right) \\
&\le \exp\left(-\frac{\lambda^2}{An\sum_{k=1}^n a_k^2(\theta_2(X_k))^2}\right)
\end{aligned}
$$

where $A = \sqrt{3.1}e^{9/8}$.

Proof. Since $\sum_{k=1}^n a_k X_k$ is sub–gaussian with norm $\tau(\sum_{k=1}^n a_k X_k)$ and

$$
\tau^2(\sum_{k=1}^n a_k X_k) \le n\sum_{k=1}^n a_k^2\tau^2(X_k),
$$

the result follows.

In Lemma 3.1, let $a_k = 1/n$, $k = 1, 2, \cdots, n$. Then,

$$
\begin{aligned}
P(n^{-1}\sum_{k=1}^{n} X_k > \lambda) = P(S_n > n\lambda) &\le \exp\left(-\frac{n^2\lambda^2}{2\tau^2(S_n)}\right) \\
&\le \exp\left(-\frac{n\lambda^2}{2\sum_{k=1}^n \tau^2(X_k)}\right) \\
&\le \exp\left(-\frac{n\lambda^2}{A\sum_{k=1}^n(\theta_2(X_k))^2}\right).
\end{aligned}
$$

The least upper bound for the above probability depends on the norm τ, but in most cases its direct calculation is difficult. Thus it would be useful to find upper bounds in terms of $\theta_2(X)$ or $\theta_1(X)$.

Corollary 3.2. Let $X_1, X_2, \cdots, X_n$ be independent random variables such that $|X_i| \le m$ and $E(X_i) = 0$, $1 \le i \le n$ and let $a_i \in R$, $1 \le i \le n$. Then for $\lambda > 0$

$$
\begin{aligned}
P(\sum_{k=1}^{n} a_k X_k > \lambda) &\le \exp\left(-\frac{\lambda^2}{2\sum_{k=1}^n a_k^2\tau^2(X_k)}\right) \\
&\le \exp\left(-\frac{\lambda^2}{2m^2\sum_{k=1}^n a_k^2}\right)
\end{aligned}
$$

Proof. Since $\sum_{k=1}^{n} a_k X_k \in \text{sub}(R)$ and $X_1, X_2, \cdots, X_n$ are independent random variables, $\tau^2(\sum_{k=1}^{n} a_k X_k) = \sum_{k=1}^{n} a_k^2 \tau^2(X_k)$. Thus the result follows from Lemma 3.1 and the basic property (3) of sub-gaussian random variables.

In Corollary 3.2, let $a_k = \frac{1}{n}$, $k = 1, 2, \cdots, n$. Then,

$$P(n^{-1} \sum_{k=1}^{n} X_k > \lambda) \leq \exp\left(-\frac{n\lambda^2}{2m^2}\right).$$

Now, we are interested in relaxing the independence assumption in Corollary 3.2. Azuma(1967) relaxed the independence assumption on $\{X_n : n = 1, 2, \cdots\}$ as follows.

Lemma 3.3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that $|X_n| \leq 1$ a.s. for all $n \geq 1$ and $E(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0$ for all $i_1 < i_2 < \cdots, i_k; k = 1, 2, \cdots$. Then, $E(\exp(\lambda \sum_{k=1}^{n} a_{nk} X_k)) \leq \exp(\lambda^2 \sum_{k=1}^{n} a_{nk}^2 / 2)$ for all $\lambda \in R$, where $\{a_{nk} : 1 \leq k \leq n, n \geq 1\}$ is a triangular array of real numbers.

Corollary 3.4. Let $X_1, X_2, \cdots, X_n$ be bounded and multiplicative random variables (that is, $|X_i| \leq m$ a.s. for all i and $E(X_{i_1} \cdots X_{i_k}) = 0$ for all $i_1 < i_2 < \cdots < i_k$ and $k = 1, 2, \cdots, n$). Then for $\lambda > 0$

$$P(\sum_{k=1}^{n} a_k X_k > \lambda) \leq \exp\left(-\frac{\lambda^2}{2m^2 \sum_{k=1}^{n} a_k^2}\right).$$

Proof. By Lemma 3.3, $\sum_{k=1}^{n} a_k X_k$ is sub-gaussian with $\tau^2(\sum_{k=1}^{m} a_k X_k) \leq m^2 \sum_{k=1}^{m} a_k^2$. Thus the result follows from Corollary 3.2.

Notice that Azuma's result can be obtained when $X' = (X_1, \cdots, X_n)$ is a sub-gaussian random vector and $B = I$. Hence we can generalize the result with sub-gaussian random vectors.

Lemma 3.5. Let X be a sub-gaussian random vector with positive definite matrix $B = (b_{ij})$. Then, for $\lambda > 0$

$$P(\sum_{k=1}^{n} a_k X_k > \lambda) \leq \exp\left(-\frac{\lambda^2}{2\alpha_n^2}\right)$$

where $\alpha_n^2 = \sum_{i=1}^n a_i^2 b_{ii} + 2\sum\sum_{i<j} a_i a_j b_{ij}$.

Proof.

$$\begin{aligned} P(\sum_{k=1}^n a_k X_k > \lambda) &= P\Big[\exp\Big(\lambda \sum_{k=1}^n a_k X_k/\alpha_n^2\Big) > \exp(\lambda^2/\alpha_n^2)\Big] \\ &\leq \exp(-\lambda^2/\alpha_n^2) E\{\exp\Big(\lambda \sum_{k=1}^n a_k X_k/\alpha_n^2\Big)\} \\ &\leq \exp(-\lambda^2/\alpha_n^2)\exp(\lambda^2/2\alpha_n^2) \\ &= \exp(-\lambda^2/2\alpha_n^2) \end{aligned}$$

where $\alpha_n^2 = \sum_{i=1}^n a_i^2 b_{ii} + 2\sum\sum_{i<j} a_i a_j b_{ij}$.

Corollary 3.6. Let $X \in \overline{\text{sub}}(R^n)$. Then for $\lambda > 0$

$$P[\sum_{k=1}^n a_k X_k > \lambda] \leq \exp\left(-\frac{\lambda^2}{2\alpha_n^2}\right)$$

where $\alpha_n^2 = \text{Var}(\sum_{k=1}^n a_k X_k) = \sum_{i=1}^n a_i^2 \sigma_i^2 + 2\sum\sum_{i<j} a_i a_j \sigma_{ij}$.

In addition to the hypotheses of Corollary 3.6, suppose that $E(X_i^2) = \sigma^2$ for all i and $E(X_i X_j) = \rho$ for all $i \neq j$. Then , for $\lambda > 0$

$$P(n^{-1}\sum_{k=1}^n X_k > \lambda) \leq \exp\left(-\frac{\lambda^2}{2\alpha_n^2}\right)$$

where $\alpha_n^2 = n^{-1}(\sigma^2 + (n-1)\rho)$.

Corollary 3.7. Let B be a symmetric positive definite matrix. If $E(X) = 0$ and $X'B^{-1}X \leq m$ a.s., then for $\lambda > 0$

$$P(\sum_{k=1}^n a_k X_k > \lambda) \leq \exp\left(-\frac{\lambda^2}{2\alpha_n^2}\right)$$

where $\alpha_n^2 = m(\sum_{i=1}^n a_i^2 b_{ii} + 2\sum\sum_{i<j} a_i a_j b_{ij})$.

Proof. By Lemma 2.3, X is a sub–gaussian random vector with the symmetric positive definite matrix mB, and hence by Lemma 3.5 the result follows.

In Corollary 3.7, since B^{-1} is positive definite, $0 < X'B^{-1}X \le \lambda_1^{-1}X'X$, where $\lambda_1(> 0)$ is the smallest eigenvalue of B. Thus, the following corollary follows.

Corollary 3.8. Let B be a symmetric positive definite matrix. If $EX = 0$ and $X'X \le m\lambda_1$ a.s., where λ_1 is the smallest eigenvalue of B. Then for $\lambda > 0$

$$P(\sum_{k=1}^{n} a_k X_k > \lambda) \le \exp\left(-\frac{\lambda^2}{2\alpha_n^2}\right)$$

where $\alpha_n^2 = m(\sum_{i=1}^n a_i^2 b_{ii} + 2\sum\sum_{i<j} a_i a_j b_{ij})$.

4. STRONG LAWS OF LARGE NUMBERS

In this section complete convergence results for sums of triangular arrays of random variables and random elements in $L^1(R)$ are obtained.

Definition. A sequence of random variables $\{X_n : n = 1, 2, \cdots\}$ is said to be strictly sub–gaussian if $(X_1, X_2, \cdots, X_k) \in \overline{\text{sub}}(R^k)$ for all $k \ge 1$.

Theorem 4.1. Let $\{X_{nk} : 1 \le k \le n, n \ge 1\}$ be a triangular array of random variables such that $EX_{nk} = 0$ for each n and k and for each $n \ge 1$, $X_n' B_n^{-1} X_n \le m$ a.s., where $X_n' = (X_{n1}, X_{n2}, \cdots, X_{nn})$ and B_n is the covariance matrix of X_n. Suppose that

(1) $\sup_{n,k} E(X_{nk}^2) < \infty$

(2) $\sup_{k \ne l} E(X_{nk}X_{nl}) = O(n^{-a})$, $a > 0$.

Then

$$n^{-d} \sum_{k=1}^{n} X_{nk} \to 0 \quad \text{completely if} \quad d > 1 - \frac{a}{2}.$$

Proof. Let $\epsilon > 0$ be given. By Corollary 3.7

$$\begin{aligned}\sum_{n=1}^{\infty} P[|n^{-d}\sum_{k=1}^{n} X_{nk}| > \epsilon] \\ &\leq 2\sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon^2 n^{2d}}{2m(\sum_{k=1}^{m}\sigma_{nk}^2 + \sum\sum_{k\neq l}\sup E(X_{nk}X_{nl}))}\right) \\ &\leq 2\sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon^2 n^{2d-1}}{2m(\sup_k \sigma_{nk}^2 + (n-1)\sup_{k\neq l} E(X_{nk}X_{nl}))}\right) \\ &< \infty\end{aligned}$$

if $d > 1 - a/2$.

Complete convergence results for sums of triangular arrays of random elements in $L^1(R)$ is obtained next. More information on laws of large numbers for random elements in $L^1(R)$ is available from Taylor and Lee(1990). Theorem 4.2 and Theorem 4.3 are motivated by the study of L^1 consistency of kernel density estimators (cf.Devroye and Gyorfi(1985)) and are patterned after similar results for $C_0(R)$ in Taylor and Hu(1987b). Recall that for a random element $\tilde{X}_{nk}$ in $L^1(R)$, X_{nk} denotes a representable member of the equivalence class $\tilde{X}_{nk}$ such that for each $t \in R$, $X_{nk}(t,\cdot)$ is an extended random variable.

Theorem 4.2. (Taylor and Lee(1990)) Let $\{\tilde{X}_{nk} : 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent random elements in $L^1(R)$ with $\widetilde{EX}_{nk} = \tilde{0}$ for each n and k and let $1/2 < d \leq 1$. Suppose that for each n

(1) $X_{nk}(t) = 0$ for $|t| > An^s$, $s \geq 0$, $A > 0$,

(2) $\sup_t |X_{nk}(t)| \leq \gamma_n$ a.s., $\gamma_n^2 = O(n^r)$, $r \geq 0$,

(3) $\sup_{u,v} |X_{nk}(u,\omega) - X_{nk}(v,\omega)| \leq C_n|u-v|^a$ a.s., where $0 < a \leq 1, C_n = O(n^\beta), \beta \geq 0$.

Then

$$\|n^{-d}\sum_{k=1}^{n}\tilde{X}_{nk}\| \to 0 \quad \text{completely if} \quad 2d > 2s + r + 1.$$

The above theorem can be applied in establishing the L^1 consistency of kernel density estimators, that is, $\int |\frac{1}{nh_n}\sum_{k=1}^{n} K(\frac{t-X_k}{h_n}) - f(t)|dt \to 0$ completely, when a random sample $X_1, \cdots, X_n$ comes from a density function $f(t)$

with $E|X_1|^r < \infty$, $r > 2$ and $K(t)$ satisfies some conditions (cf.Taylor and Lee(1990)). The independence assumption in Theorem 4.2 can be relaxed as follows.

Theorem 4.3. Let the triangular array of random elements $\{\tilde{X}_{nk} : 1 \leq k \leq n, n \geq 1\}$ in $L^1(R)$ be rowwise strictly sub-gaussian and let $1/2 < d \leq 1$. Suppose that for each n

(1) $X_{nk}(t) = 0$ for $|t| > An^s$,

(2) $\sup_{u,v} |X_{nk}(u,\omega) - X_{nk}(v,\omega)| \leq C_n |u-v|^a$ a.s., $C_n = O(n^b)$, $a > 0$, $b > 0$.

Then

$$\|n^{-d}\sum_{k=1}^{n} \tilde{X}_{nk}\| \to 0 \quad \text{completely if} \quad (\sigma_n^2 + (n-1)\rho_n)n^{1+2s-2d} = O(n^{-r})$$

where $\sigma_n^2 = \sup_{k,t} E(X_{nk}(t))^2$, $\rho_n = \sup_{k,l,t} E(X_{nk}(t)X_{nl}(t))$ and $r > 0$.

Proof. Let $\epsilon > 0$ be given. For each n, let $\{t_0, t_1, \cdots, t_{m_n}\}$ be a partition over $[-An^s, An^s]$ with equal length Δt,

$$\Delta t = \epsilon^{1/a}(4AC_n n^{1-d+s})^{-1/a}.$$

Let $Y_{nk}(\omega) = \sum_{i=1}^{m_n} X_{nk}(t_i,\omega) I_{(t_{i-1},t_i]}(t)$ for each $\omega \in \Omega$. Then by (2) and Corollary 3.6,

$$\begin{aligned}
P[\|n^{-d}\sum_{k=1}^{n}\tilde{X}_{nk}\| > \epsilon] &= P[\|n^{-d}\sum_{k=1}^{n} X_{nk}\| > \epsilon] \\
&\leq P[\|n^{-d}\sum_{k=1}^{n}(X_{nk} - Y_{nk})\| > \frac{\epsilon}{2}] + P[\|n^{-d}\sum_{k=1}^{n} Y_{nk}\| > \frac{\epsilon}{2}] \\
&\leq \sum_{k=1}^{n} P[\|X_{nk} - Y_{nk}\| > \frac{\epsilon n^{d-1}}{2}] + P[\|\sum_{k=1}^{n} Y_{nk}\| > \frac{\epsilon n^d}{2}] \\
&\leq \sum_{k=1}^{n} P[\sum_{i=1}^{m_n}\int |X_{nk}(t) - X_{nk}(t_i)| I_{(t_{i-1},t_i]}(t)dt > \frac{\epsilon n^{d-1}}{2}] \\
&\quad + \sum_{i=1}^{m_n} P[\int |\sum_{k=1}^{n} X_{nk}(t_i) I_{(t_{i-1},t_i]}(t)|dt > \frac{n^d \epsilon}{2m_n}] \\
&= \sum_{k=1}^{n} P[m_n C_n (\Delta t)^{1+a} > \frac{\epsilon n^{d-1}}{2}]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m_n} P[|\sum_{k=1}^{n} X_{nk}(t_i)| > 4^{-1}A^{-1}\epsilon n^{d-s}] \\
&\le 2m_n \exp\left(-\frac{\epsilon^2 n^{2d-2s-1}}{32A^2(\sigma_n^2+(n-1)\rho_n)}\right) \\
&= (4A)^{1+1/a}\epsilon^{-1/a}C_n^{1/a}n^{s+(1-d+s)/a}\exp\left(-\frac{\epsilon^2 n^{2d-2s-1}}{32A^2(\sigma_n^2+(n-1)\rho_n)}\right).
\end{aligned}
$$

Hence, it follows from the integral test that

$$\sum_{n=1}^{\infty} P[\|n^{-d}\sum_{k=1}^{n}\tilde{X}_{nk}\| > \epsilon] < \infty$$

since $n^{2d-2s-1}(\sigma_n^2+(n-1)\rho_n)^{-1} \ge Kn^r$, $r > 0$ and $K > 0$.

Corollary 4.4. Let $\{\tilde{X}_{nk} : 1 \le k \le n, n \ge 1\}$ be a triangular array of random elements in $L^1(R)$ with $\widetilde{EX}_{nk} = \tilde{0}$ for each n and k and let $1/2 < d \le 1$. Suppose that for each n

(1) $X_{nk}(t) = 0$ for $|t| > An^s$,

(2) $\sup_t X_n(t)'B_n(t)^{-1}X_n(t) \le \gamma$ a.s., where $X_n' = (X_{n1},\cdots,X_{nn})$ and $B_n(t)$ is the covariance matrix of $X_n(t)$,

(3) $\sup_{u,v}|X_{nk}(u,\omega)-X_{nk}(v,\omega)| \le C_n|u-v|^a$ a.s., $C_n = O(n^b)$, $a > 0$, $b > 0$.

Then

$$\|n^{-d}\sum_{k=1}^{n}\tilde{X}_{nk}\| \to 0 \quad \text{completely if} \quad (\sigma_n^2+(n-1)\rho_n)n^{1+2s-2d} = O(n^{-r})$$

where $\sigma_n^2 = \sup_{k,t}E(X_{nk}(t))^2$, $\rho_n = \sup_{k,l,t}E(X_{nk}(t)X_{nl}(t))$ and $r > 0$.

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