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The Asymptotic Unbiasedness of S^2 in the Linear Regression Model with Moving Average or Particular S-th Order Autocorrelated Disturbances

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ABSTRACT

The OLS-estimator of the disturbance variance in the linear regression model is shown to be asymptotically unbiased when the disturbances are $MA(1)$-process or particular s-th order autocorrelated $AR(s)$-process.

KEYWORDS : Ordinary least square estimator, Linear regression, Disturbance variance, Asymptotic unbiasedness.

1. INTRODUCTION

We consider the standard linear regression model

$$Y = X\beta + u, \tag{1.1}$$

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where Y is the $T \times 1$ vector of observations, X is $T \times k$ non-stochastic regressor matrix, β is $k \times 1$ unknown parameter vector, and u is $T \times 1$ disturbance vector with expectation $E(u) = 0$ and covariance matrix $E(uu') = \sigma_u^2 V$, where V is assumed to be symmetric and positive definite.

It is well known that the OLS (Ordinary Least Square)-based estimator of σ^2,

$$S^2 = \frac{1}{T-k}(\hat{u}'\hat{u}) = \frac{1}{T-k}(Y - X\hat{\beta})'(Y - X\hat{\beta}), \tag{1.2}$$

where $\hat{\beta} = (X'X)^{-1}X'Y$, is a biased estimator of σ^2, when $V \neq I$.

An important question, in this context, is the properties of the biased estimator of S^2 when the disturbances are correlated. Some results on the bias, in case of the first-order Autoregressive, $AR(1)$-disturbances, are given in Sathe and Vinod (1974), and Neudecker (1977, 1978). Sathe and Vinod tabulate upper and lower bounds on the bias for a variety of values for T, k and ρ, where ρ is the coefficient of the $AR(1)$-process. Neudecker (1977, 1978) provides bounds for the relative bias $E(S^2/\sigma^2)$ which for given ρ become narrower as the sample size increases, but do not seem to converge to unity. Krämer (1991) has shown the asymptotic unbiasedness of S^2 for the $AR(1)$-disturbance.

In this note, I will show the asymptotic unbiasedness of S^2 if u is a first-order Moving-Average process, $MA(1)$, or a particular s-th order autocorrelated process, $AR(s)$.

2. ASYMPTOTIC UNBIASEDNESS OF S^2

2.1. Moving average disturbances

Let u be generated by a invertible $MA(1)$ process such as

$$u_t = \varepsilon_t - \theta\varepsilon_{t-1}, \qquad t = 1, 2, \cdots, T, \tag{2.1}$$

where $|\theta| < 1$ and $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = \sigma_\varepsilon^2$.

The $T \times T$ autocovariance matrix is given by

$$E(uu') = \sigma_\varepsilon^2 V, \tag{2.2}$$

where σ_ε^2 is $\sigma_u^2/(1+\theta^2)$ and the form of V is three diagonal matrix having $(1+\theta^2)$ on the main diagonal and $(-\theta)$ on the adjacent diagonals

The characteristic roots of V in (2.2) can be obtained from Graybill (1983, p. 284 Theorem 8.15.3) as

$$\lambda_i = \frac{1+\theta^2+2|\theta|\cos(\frac{i\pi}{T+1})}{1+\theta^2}, \quad i = 1, 2, \cdots, T, \tag{2.3}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_T$ are ordered characteristic roots.

An upper bound for the numerator of (2.3) is given by

$$1 + 2|\theta|\cos(\frac{\pi i}{T+1}) + \theta^2 \leq (1+|\theta|)^2, \tag{2.4}$$

which implies that

$$\lambda_i = \frac{1+|\theta^2|+2\theta\cos(\frac{i\pi}{T+1})}{1+\theta^2} \leq \frac{(1+|\theta|)^2}{1+\theta^2}, \tag{2.5}$$

irrespective of sample size T.

From Sathe and Vinod (1974) and Dufour (1986) we have the inequalities

$$0 \leq \frac{\text{mean of } T-k}{\text{smallest roots of } V} \leq E(\frac{S^2}{\sigma^2}) \leq \frac{\text{mean of } T-k}{\text{greatest roots of } V} \leq \frac{T}{T-k}, \tag{2.6}$$

which implies that the upper bound for $E(\frac{S^2}{\sigma^2})$ tends to one as $T \to \infty$.

It remains to show that the lower bound tends to one as well.

A lower bound for the mean of the $T-k$ smallest characteristic roots of V may be derived as follows:

$$\frac{1}{T-k}\sum_{i=1}^{T-k}\lambda_{i+k} = \frac{1}{T-k}(\sum_{i=1}^{T}\lambda_i - \sum_{i=1}^{k}\lambda_i) = \frac{1}{T-k}(tr(V) - \sum_{i=1}^{k}\lambda_i)$$

$$\geq \frac{T}{T-k} - \frac{k}{T-k}(\frac{(1+|\theta|)^2}{1+\theta^2}), \tag{2.7}$$

from (2.5). Obviously, the first term on the right hand side in (2.7) tends to one and the second term to zero as $T \to \infty$. Thus S^2 is asymptotically unbiased for σ^2, regardless of the regressor matrix X.

2.2. Particular s-th order autocorrelated disturbances

Thomas and Wallis (1971), Wallis (1972) and King (1984) considered the s-th order autocorrelated disturbances model which is given as:

$$u_t = \rho u_{t-s} + \varepsilon_t, \tag{2.8}$$

where s denotes the "seasons" per year, $|\rho| < 1$ and the $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with a mean 0 and a variance σ_ε^2. For observations of m years, $T = sm$. The autocovariance $E(u_t, u_{t-j})$ is zero unless j is an integer multiple of s, in which case $E(u_t, u_{t-j}) = \sigma_u^2 \rho^{j/s}$, where σ_u^2 is $\sigma_\varepsilon^2/(1-\rho^2)$.

Thus we have

$$E(uu') = \sigma_u^2 V_s = \sigma_u^2 (V_1 \otimes I_s), \tag{2.9}$$

where $V_1 = (V_{ij}) = \rho^{|i-j|}, \quad i,j = 1,2,\cdots,m, \quad I_s$ is the $S \times S$ identity matrix and $\otimes$ denotes the Kronecker product.

The characteristic roots of V_1 are given by (see Neudecker(1977), p. 1258)

$$\lambda_i \cong \frac{1-\rho^2}{1 - 2\rho\cos\theta_{m+1-i} + \rho^2}, \qquad i = 1,2,\cdots,m, \tag{2.10}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ are ordered characteristic roots and the $cos\theta$'s are the roots of a certain mth degree polynomial in $cos\theta$, namely

$$\frac{sin(m+1)\theta}{sin\theta} - 2\rho\frac{sin(m)\theta}{sin\theta} + \rho^2\frac{sin(m-1)\theta}{sin\theta} = 0. \tag{2.11}$$

Since the characteristic roots of a Kronecker product of matrices are given by the product of the characteristic roots of these matrices, (see Horn and Johnson(1990), p. 245), we obtain the characteristic roots of V_s as the products of the characteristic roots of V_1 and the characteristic roots of I_s. Since the characteristic roots of I_1 are 1's. The characteristic roots of V_s are equal to those of V_1 with multiplicity s. Therefore in V_s each of them appear with multiplicity s such as

$$\lambda_i \cong \frac{1-\rho^2}{1-2\rho\cos\theta_{m-[\frac{i-1}{s}]}+\rho^2}, \qquad i=1,2,\cdots,T, \tag{2.12}$$

where $[f]$ denotes the integer part of the real quantity f.

Hence we can observe that the characteristic roots of V_s are as follows.

$$\lambda_1=\cdots=\lambda_s \geq \lambda_{s+1}=\cdots=\lambda_{2s}\geq\cdots\ \cdots\geq\lambda_{(m-1)s+1}=\cdots=\lambda_{ms}. \tag{2.13}$$

An upper bound for the denominator of (2.12) is given by

$$1-2\rho\cos\theta_{m-[\frac{i-1}{s}]}+\rho^2 \geq (1-|\rho|)^2, \tag{2.14}$$

which implies that

$$\lambda_i \leq \frac{1-\rho^2}{(1-|\rho|)^2}, \tag{2.15}$$

irrespective of sample size T.

In context of (2.6) we consider the lower bound for the mean of the $T-k$ smallest characteristic roots of V_s as follows:

$$\frac{1}{T-k}\sum_{i=1}^{T-k}\lambda_{i+k}=\frac{T}{T-k}-\frac{1}{T-k}\sum_{i=1}^{k}\lambda_i \geq \frac{T}{T-k}-\frac{k}{T-k}\left(\frac{1-\rho^2}{(1-|\rho|)^2}\right), \tag{2.16}$$

from (2.15). The first term (2.16) on the right tends to one and the second term tends to zero as $T\to\infty$. Therefore S^2 is asymptotically unbiased of σ^2, regardless of the regressor matrix X.

3. CONCLUSION AND REMARKS

It is shown in this note that S^2 is asymptotic unbiased in the linear regression model with $MA(1)$ or particular s-th order autocorrelated disturbances. Further studies of the cases of the asymptotic unbiasedness of S^2 in the more general settings such as $ARMA(p,q)$ processes are currently under investigation.

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