Journal of the Korean
Statistical Society
Vol. 23, No. 1, 1994

Characterization of the Asymptotic Distributions of Certain Eigenvalues in a General Setting

Changha Hwang[1]

ABSTRACT

Let $A(n)$ and $B(n)$ be sequences of $m \times m$ random matrices with a joint asymptotic distribution as $n \to \infty$. The asymptotic distribution of the ordered roots of $\mid A(n) - f\ B(n) \mid = 0$ depends on the multiplicity of the roots of a determinantal equation involving parameter roots. This paper treats the asymptotic distribution of the roots of the above determinantal equation in the case where some of parameter roots are zero. Furthermore, we apply our results to deriving the asymptotic distributions of the eigenvalues of the MANOVA matrix in the noncentral case when the underlying distribution is not multivariate normal and some parameter roots are zero.

1. INTRODUCTION

In many problems in multivariate statistical analysis eigenvalues of one random symmetric matrix in the metric of another are very often used. Let $A(n)$ and $B(n)$ be sequences of $m \times m$ random symmetric matrices indexed

[1] Department of Computer Science and Statistics, Kyungsung University, Pusan, 608-736, Korea.

by n, where $B(n)$ is positive definite with probability one for all n. Then, we are interested in the determinantal equation

$$| A(n) - f\ B(n) | = 0 \tag{1}$$

with m real roots with probability one. Amemiya (1986, 1990) characterized the asymptotic distribution of the ordered m roots of (1) under the case where all parameter roots are different from zero, i.e., $D(n) =$ *block diagonal* $\{d_1(n)I_{t_1}, d_2(n)I_{t_2}, \ldots, d_s(n)I_{t_s}\}$, where $\sum_{j=1}^{s} t_j = m$. In this paper, we will treat characterizing the asymptotic distribution of the roots of (1) under the case where some of parameter roots are zero, i.e., $D(n) =$ *block diagonal* $\{d_1(n)I_{t_1}, d_2(n)I_{t_2}, \ldots, d_s(n)I_{t_s}, \mathbf{0}\}$, where $\sum_{j=1}^{s} t_j = k$. Thus our results will be an extension of Amemiya (1990). Furthermore, we will apply our results to deriving the asymptotic distribution of the eigenvalues of the noncentral MANOVA matrix for nonnormal populations when some of parameter roots are zero. Hsu (1941) and Anderson (1951) obtained the asymptotic distribution of the roots of (1) where $A(n)$ is a noncentral Wishart matrix with fixed degrees of freedom and $B(n)$ is a Wishart matrix with increasing degrees of freedom. Bai (1984) presented a lemma to fill a gap which existed in the proofs of Hsu (1941), Anderson (1951) and subsequent papers. In multiple discriminant analysis the procedures commonly used to determine the number of useful discriminant functions required to describe group differences involve testing a sequence of dimensionality hypotheses as well as model fitting approaches based on Akaike's method, Mallows' method and Schwarz's method. All of the available methods involve eigenvalues of the MANOVA matrix. Thus in order to evaluate the performance of various methods for nonnormal populations we need to know the asymptotic distributions of the eigenvalues which will be derived in Theorem 2. See, for details, Muirhead (1982) and Hwang (1991).

For the special case with $B(n) = I_m$ for all n, Anderson (1963) derived the asymptotic distribution of the roots for a normal population, and Waternaux (1976), Davis (1977), Fujikoshi (1980) and Fang and Krishnaih (1982) considered nonnormal populations.

2. MAIN RESULT

Amemiya (1990) presented the following result using the lemma presented by Bai (1984). Bai's lemma was also reported in Anderson (1989) and Amemiya (1990). A sketch of proof of Amemiya's result is given below. Basically Bai's lemma and Rubin's theorem are used to prove this result and Theorem 1. This result uses the following notations and assumptions. Let

$$D(n) = block\ diagonal\ \{d_1(n)I_{t_1}, d_2(n)I_{t_2}, \ldots, d_s(n)I_{t_s}\}$$

be a nonstochastic $m \times m$ diagonal matrix, where $\sum_{j=1}^{s} t_j = m$. Assume that as $n \to \infty, d_j(n) \to d_j,\ j = 1, 2, \ldots, s$, where $d_1 > \cdots > d_s$. We assume that as $n \to \infty$,

$$\sqrt{n}[\, A(n) - D(n), B(n) - I_m \,] \xrightarrow{L} [\, U, V \,],$$

where

$$U = \begin{bmatrix} U_{11} & \cdots & U_{1s} \\ \vdots & & \vdots \\ U_{s1} & \cdots & U_{ss} \end{bmatrix}, \ V = \begin{bmatrix} V_{11} & \cdots & V_{1s} \\ \vdots & & \vdots \\ V_{s1} & \cdots & V_{ss} \end{bmatrix},$$

U_{ij} and V_{ij} are $t_i \times t_j$. Let $f_1 \geq \cdots \geq f_m$ be the roots of (1). We consider this as the case where $A(n)$ and $B(n)$ are already transformed to a canonical form so that the 'true (population) matrices' corresponding to $A(n)$ and $B(n)$ are diagonal. Define $p_0 = 0,\ p_j = \sum_{l=1}^{j} t_l,\ j = 1, 2, \ldots, s-1,\ p_s = m$,

$$\mathbf{g}_j(n) = \sqrt{n}\left[\, f_{p_{j-1}+1} - d_j(n), f_{p_{j-1}+2} - d_j(n), \ldots, f_{p_j} - d_j(n)\right],\ j = 1, 2, \ldots, s,$$

$$\mathbf{h}(n) = [\, \mathbf{g}_1(n),\ \mathbf{g}_2(n), \ldots, \mathbf{g}_s(n)\,].$$

Let $h_i(n)$ and h_i be the i^{th} elements of $\mathbf{h}(n)$ and $\mathbf{h}$, respectively. The $h_i(n)$'s are functions of the elements of random matrices $U(n) = \sqrt{n}\ [A(n) - D(n)]$ and $V(n) = \sqrt{n}\ [\, B(n) - I_m\,]$ where the function depends on n and a nonstochastic matrix $D(n)$. To use Rubin's theorem, we treat $\mathbf{h}(n)$, $\mathbf{h}$, $U(n)$, $V(n)$, U and V as nonstochastic vectors and matrices and then show that if

$$U(n) \to U,\ V(n) \to V,\ D(n) \to D,\ n \to \infty, \tag{2}$$

then $\mathbf{h}(n) \to \mathbf{h}$. The f_i's are the real roots of the polynomial equation (1), and are continuous functions of the elements of $A(n)$ and $B(n)$. Since $A(n) \to D$ and $B(n) \to I_m$, $f_i \to d_i$, $i = 1, \ldots, m$. We partition $U(n)$ and $V(n)$ corresponding to U and V, respectively. Note that (1) is equivalent to the following determinant:

$$\det \begin{bmatrix} C_{11}(n) & \cdots & C_{1j}(n) & \cdots & C_{1s}(n) \\ & & \vdots & & \\ C_{j1}(n) & \cdots & C_{jj}(n) & \cdots & C_{js}(n) \\ & & \vdots & & \\ C_{s1}(n) & \cdots & C_{sj}(n) & \cdots & C_{ss}(n) \end{bmatrix} = 0, \tag{3}$$

where

$$C_{ii}(n) = (d_i(n) - f)I_{t_i} + \tfrac{1}{\sqrt{n}}(U_{ii}(n) - fV_{ii}(n)), \text{ for } i = 1, \ldots, s,$$

$$C_{ij}(n) = \tfrac{1}{\sqrt{n}}\,(U_{ij}(n) - fV_{ij}(n)), \text{ for } i, j = 1, \ldots, s \ (i \neq j).$$

Take the variable transformation $f = d_j(n) + \xi(n)\frac{1}{\sqrt{n}}$ in (3). Multiply by $n^{\frac{1}{4}}$ rows containing $t_j \times t_1$ submatrix C_{j1} and columns containing $t_1 \times t_j$ submatrix C_{1j} of the determinant on the left hand side of (3) and make n tend to infinity. Note that the $t_j \times t_j$ submatrix is multiplied by $n^{\frac{1}{4}}$ twice. f_i's are also the roots of the following determinantal equation:

$$\det \begin{bmatrix} C_{11}(n) & \cdots & n^{\frac{1}{4}}C_{1j}(n) & \cdots & C_{1s}(n) \\ & & \vdots & & \\ n^{\frac{1}{4}}C_{j1}(n) & \cdots & \sqrt{n}C_{jj}(n) & \cdots & n^{\frac{1}{4}}C_{js}(n) \\ & & \vdots & & \\ C_{s1}(n) & \cdots & n^{\frac{1}{4}}C_{sj}(n) & \cdots & C_{ss}(n) \end{bmatrix} = 0. \tag{4}$$

If we denote by $\xi_i(n) = \sqrt{n}\,[f_i - d_j(n)]$ the roots of (3) after variable transformation $\xi(n) = \sqrt{n}\,[f - d_j(n)]$, we know that

$$\xi_i(n) \to \begin{cases} +\infty, & i = 1, 2, \ldots, p_{j-1}, \\ -\infty, & i = p_j + 1, \ldots, m, \end{cases}$$

since $d_j(n) \to d_j$ and $f_i \to d_j$, $i = p_{j-1}+1, \ldots, p_j$. Hence by Bai's lemma we prove that

$$\xi_i(n) \to \xi_i,\ i = p_{j-1}+1, \ldots, p_j,$$

where ξ_i, $i = p_{j-1}+1, \ldots, p_j$ are roots of

$$\det\,[\, U_{jj} - d_j V_{jj} - \xi I_{t_j} \,] = 0.$$

Note that

$$\xi_i(n) = h_i(n),\ \ p_{j-1}+1 \le i \le p_j$$

and that

$$\xi_{p_{j-1}+1}(n) \ge \xi_{p_{j-1}+2}(n) \ge \cdots \ge \xi_{p_j}(n).$$

Thus, nonstochastically, $\mathbf{h}(n) \to \mathbf{h} = (\ \mathbf{g}_1,\ \mathbf{g}_2, \ldots, \mathbf{g}_s\ )$ under (2), where $\mathbf{g}_j$, $1\times t_j$, is the vector of the ordered eigenvalues of $U_{jj} - d_j V_{jj}$, i.e., the $t_j \times t_j$ submatrix of $U - DV$, where $D = block\ \ diagonal\ \{\ d_1 I_{t_1},\ d_2 I_{t_2}, \ldots, d_s I_{t_s}\ \}$. Because the elements of $\mathbf{h}$ are continuous functions of the elements of U and V, Rubin's theorem applies and $\mathbf{h}(n) \xrightarrow{L} \mathbf{h}$. Now we will consider the asymptotic distribution of roots of (1) for the case where some of diagonal elements of $D(n)$ are zero, i.e.,

$$D(n) = block\ \ diagonal\ \{\ d_1(n) I_{t_1}, d_2(n) I_{t_2}, \ldots, d_s(n) I_{t_s}, \mathbf{0}\ \},$$

where $\sum_{j=1}^{s} t_j = k$. This case is an extension of Amemiya (1990) in some sense. Again we assume that as $n \to \infty$, $d_j(n) \to d_j$, $j = 1, \ldots, s$, where $d_1 > \cdots > d_s$. Let $D = block\ diag\ \{\ d_1 I_{t_1}, d_2 I_{t_2}, \ldots, d_s I_{t_s}, \mathbf{0}\ \}$. Put

$$A(n) = \begin{bmatrix} A_{11}(n) & A_{12}(n) & \cdots & A_{1s}(n) & A_{1,k+1}(n) \\ A_{21}(n) & A_{22}(n) & \cdots & A_{2s}(n) & A_{2,k+1}(n) \\ & & \vdots & & \\ A_{s1}(n) & A_{s2}(n) & \cdots & A_{ss}(n) & A_{s,k+1}(n) \\ A_{k+1,1}(n) & A_{k+1,2}(n) & \cdots & A_{k+1,s}(n) & A_{k+1,k+1}(n) \end{bmatrix}$$

and let $\tilde{A}(n)$ be the matrix derived from $A(n)$ by replacing $A_{k+1,k+1}(n)$ in $A(n)$ by $\sqrt{n}A_{k+1,k+1}(n)$. We note that $\tilde{A}(n)$ is a symmetric matrix. Assume that as $n \to \infty$,

$$\sqrt{n}[\tilde{A}(n) - D(n),\ B(n) - I_m] \xrightarrow{L} [\,U,\ V\,],$$

$$U = \begin{bmatrix} U_{11} & \cdots & U_{1s} & U_{1,k+1} \\ & \vdots & & \\ U_{s1} & \cdots & U_{ss} & U_{s,k+1} \\ U_{k+1,1} & \cdots & U_{k+1,s} & U_{k+1,k+1} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & \cdots & V_{1s} & V_{1,k+1} \\ & \vdots & & \\ V_{s1} & \cdots & V_{ss} & V_{s,k+1} \\ V_{k+1,1} & \cdots & V_{k+1,s} & V_{k+1,k+1} \end{bmatrix},$$

U_{ij} and V_{ij} are $t_i \times t_j$, $i,j = 1,\ldots,s$, $U_{i,k+1}$ and $V_{i,k+1}$ are $t_i \times (m-k)$, $i = 1,\ldots,s$, $U'_{i,k+1} = U_{k+1,i}$, $V'_{i,k+1} = V_{k+1,i}$, and $U_{k+1,k+1}$ and $V_{k+1,k+1}$ are $(m-k) \times (m-k)$.

Theorem 1. Define $p_0 = 0$, $p_j = \sum_{l=1}^{j} t_l$, $j = 1,2,\ldots,s-1$, $p_s = k$,

$$\mathbf{g}_j(n) = \sqrt{n}\left[\, f_{p_{j-1}+1} - d_j(n), f_{p_{j-1}+2} - d_j(n), \ldots, f_{p_j} - d_j(n)\right],\ j = 1,2,\ldots,s,$$

$$\mathbf{h}(n) = [\,\mathbf{g}_1(n),\ \mathbf{g}_2(n),\ldots,\mathbf{g}_s(n)\,],$$

$$\mathbf{f}(n) = n\ (\ f_{k+1},\ldots,f_m\ ).$$

Then, as $n \to \infty$,

$$\mathbf{h}(n) \xrightarrow{L} \mathbf{h} = (\ \mathbf{g}_1,\ \mathbf{g}_2,\ldots,\mathbf{g}_s\ ),$$

where $\mathbf{g}_j$, $1 \times t_j$, is the vector of the ordered eigenvalues of $U_{jj} - d_jV_{jj}$. And the asymptotic distribution of $\mathbf{f}(n)$ is the same as the distribution of the roots of a $(m-k) \times (m-k)$ matrix

$$U_{k+1,k+1} - \sum_{i=1}^{s} \frac{1}{d_i} U'_{i,k+1}U_{i,k+1}.$$

Proof. This theorem is proved in the similar way. Since the asymptotic distribution of $\mathbf{h}(n)$ is proved in the same way as before, we will only prove the asymptotic distribution of $\mathbf{f}(n)$, using Bai's lemma and Rubin's theorem. The nf_i's are functions of the elements of random matrices $U(n) = \sqrt{n}\,[\tilde{A}(n) - D(n)]$ and $V(n) = \sqrt{n}\,[\,B(n) - I_m\,]$ where the function depends on n and a nonstochastic matrix $D(n)$. To use Rubin's theorem, we treat $\mathbf{f}(n)$, $U(n)$, $V(n)$, U and V as nonstochastic vectors and matrices and then show that if

$$U(n) \to U,\ V(n) \to V,\ D(n) \to D,\ n \to \infty,$$

then $\mathbf{f}(n)$ converges to the roots of a $(m-k) \times (m-k)$ matrix

$$U_{k+1,k+1} - \sum_{i=1}^{s} \frac{1}{d_i} U'_{i,k+1} U_{i,k+1}.$$

The f_i's are the real roots of the polynomial equation (1), and are continuous functions of the elements of $A(n)$ and $B(n)$. Since $A(n) \to D$ and $B(n) \to I_m$, $f_i \to d_i$, $i = 1, \ldots, m$. We partition $U(n)$ and $V(n)$ corresponding to U and V, respectively. Note that (1) is equivalent to the following determinant:

$$\det \begin{bmatrix} C_{11}(n) & C_{12}(n) & \cdots & C_{1s}(n) & C_{1,k+1}(n) \\ C_{21}(n) & C_{22}(n) & \cdots & C_{2s}(n) & C_{2,k+1}(n) \\ & & \vdots & & \\ C_{s1}(n) & C_{s2}(n) & \cdots & C_{ss}(n) & C_{s,k+1}(n) \\ C_{k+1,1}(n) & C_{k+1,2}(n) & \cdots & C_{k+1,s}(n) & C_{k+1,k+1}(n) \end{bmatrix} = 0, \quad (5)$$

where for $i, j = 1, \ldots, s$, C_{ij} are defined in the same way as before and

$$C_{i,k+1}(n) = \frac{1}{\sqrt{n}}\,(U_{i,k+1}(n) - fV_{i,k+1}(n)), \text{ for } i = 1, \ldots, s,$$

$$C_{i,k+1}(n) = C_{k+1,i}(n)',$$

$$C_{k+1,k+1}(n) = \frac{1}{n} U_{k+1,k+1}(n) - fI_{m-k} - \frac{1}{\sqrt{n}} fV_{k+1,k+1}(n).$$

If we denote by $f_i = \frac{1}{n}\xi_i(n)$ the roots of (5) after variable transformation $f = \frac{1}{n}\xi(n)$, multiplying by $\sqrt{n}$ the last $m-k$ rows and columns of the determinant in (5), and making n tend to infinity, we get the following limiting equation:

$$\det \begin{bmatrix} d_1 I_{t_1} & \cdots & 0 & U_{1,k+1} \\ & \vdots & & \\ 0 & \cdots & d_s I_{t_s} & U_{s,k+1} \\ U_{k+1,1} & \cdots & U_{k+1,s} & U_{k+1,k+1} - \xi I_{m-k} \end{bmatrix} = 0 \tag{6}$$

The equation (6) is equivalent to

$$\det \left[U_{k+1,k+1} - \sum_{i=1}^{s} \frac{1}{d_i} U'_{i,k+1} U_{i,k+1} - \xi I_{m-k} \right] = 0 \tag{7}$$

We also know that

$$\xi_i(n) = n f_i \to +\infty, \qquad i = 1, \ldots, k.$$

Hence by Bai's lemma this proves that

$$\xi_i(n) = n f_i \to \xi_i, \qquad i = k+1, \ldots, m,$$

where $\xi_i, \quad i = k+1, \ldots, m$ are the roots of (7). (Note: $U_{i,k+1} = U'_{k+1,i}, \ 1 \leq i \leq s$.) Hence nonstochastically, $\mathbf{f}(n)$ converges to the roots of (7). Because the roots of (7) are continuous functions of the elements of U and V, Rubin's theorem applies and $\mathbf{f}(n)$ converges in distribution to the roots of (7). ■

We notice that we characterize the asymptotic distribution of the roots of (1) corresponding to zero parameter root.

3. APPLICATION

Hsu (1941) and Anderson (1951) derived the asymptotic distribution of the eigenvalues of the MANOVA matrix in the noncentral case when the sample size tends to infinity and the underlying distribution is multivariate normal. Now we apply Theorem 1 to extending the result of Hsu and Anderson to the case when the underlying distribution is not multivariate normal. To address this theoretically, we will consider asymptotic distribution theory in a limit where the sample sizes $q_1, \ldots, q_p$ satisfy $\frac{q_i}{n} = k_i + o(n^{-\frac{1}{2}})$

where $n = \sum_{i=1}^{p} q_i$, $k_i > 0$ and $\sum_{i=1}^{p} k_i = 1$. Put $n_2 = n - p$. In this limit, $\frac{1}{n_2}\Omega = \Theta + o(n^{-\frac{1}{2}})$ and using the transformation to canonical form we may assume that Θ is diagonal, i.e., $\Theta = diag\{\theta_1, \theta_2, \ldots, \theta_m\}$.

Theorem 2. For each $i = 1, \ldots, p$, let $\mathbf{y}_{ij} : m \times 1, j = 1, \ldots, q_i$ be a sequence of i.i.d. random vectors with $\mathbb{E}(\mathbf{y}_{ij}) = \mu_i$, covariance matrix Σ and finite fourth moments. Suppose that the p sequences are independent. Assume $\frac{q_i}{n} = k_i + o(n^{-\frac{1}{2}})$ where $k_i > 0$ and $\sum_{i=1}^{p} k_i = 1$. Then, when $H_k : \theta_1 > \cdots > \theta_k > \theta_{k+1} = \cdots = \theta_m = 0$ is true, $\sqrt{n_2}(f_r - \theta_r)$ has the asymptotic distribution

$$N(0, 4\theta_r + \theta_r^2 \sum_{i=1}^{p} k_i[\mathbb{E}(\epsilon_{i1r}^4) - 1] - 4\theta_r \sum_{i=1}^{p} k_i(\mu_{ir} - \bar{\mu}_{.r})\mathbb{E}(\epsilon_{i1r}^3))$$

for $r = 1, \ldots, k$. Also $\sqrt{n_2}(f_r - \theta_r)$ and $\sqrt{n_2}(f_l - \theta_l)$ are asymptotically dependent, for $r, l = 1, \ldots, k, (r \neq l)$. But $(\sqrt{n_2}(f_1 - \theta_1), \ldots, \sqrt{n_2}(f_k - \theta_k))$ and $(n_2 f_{k+1}, \ldots, n_2 f_m)$ are asymptotically dependent. The asymptotic distribution of $(n_2 f_{k+1}, \ldots, n_2 f_m)$ is the same as the distribution of the roots of a $(m-k) \times (m-k)$ matrix having the $W_{m-k}(n_1 - k, I_{m-k})$ distribution, where $n_1 = p - 1$.

Proof. Let $\bar{\mathbf{y}}_i = \frac{1}{q_i}\sum_{j=1}^{q_i} \mathbf{y}_{ij}$ and $\bar{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^{p} q_i\bar{\mathbf{y}}_i$, so that $\bar{\mathbf{y}}_i$ is the sample mean of the q_i observations in the i^{th} sample $(i = 1, \ldots, p)$ and $\bar{\mathbf{y}}$ is the sample mean of all observations. Let $\Omega = \Sigma^{-1} \sum_{i=1}^{p} q_i(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})'$ with $\bar{\mu} = \frac{1}{n} \sum_{i=1}^{p} q_i\mu_i$.

It is of no loss of generality to assume $\Sigma = I_m$ and $\frac{1}{n_2}\Omega = \Theta + o(n^{-\frac{1}{2}})$ where Θ is a diagonal matrix. Since the term $o(n^{-\frac{1}{2}})$ does not affect the asymptotic distribution, it is reasonable and convenient to take $\Omega = n_2\Theta$, where Θ is a fixed diagonal matrix. For our purpose, we will find the asymptotic distributions of the eigenvalues f_i's of AB^{-1} under H_k. So the parameter matrix Θ will be given by

$$\Theta = diag\{\theta_1, \ldots, \theta_k, 0, \ldots, 0\},$$

where $\theta_1, \ldots, \theta_k$ are distinct. Since $|A - fB| = 0 = |\frac{1}{n_2} A - f\frac{1}{n_2} B|$, we will consider $|\frac{1}{n_2} A - f\frac{1}{n_2} B| = 0$. Note that

$$\begin{aligned}\frac{1}{n_2} A &= \frac{1}{n_2} \sum_{i=1}^{p} q_i(\bar{\mathbf{y}}_i - \bar{\mathbf{y}})(\bar{\mathbf{y}}_i - \bar{\mathbf{y}})' \\ &= \Theta + \frac{1}{n_2} \sum_{i=1}^{p} q_i(\bar{\epsilon}_i - \bar{\epsilon})(\bar{\mu}_i - \bar{\mu})' + \frac{1}{n_2} \sum_{i=1}^{p} q_i(\bar{\mu}_i - \bar{\mu})(\bar{\epsilon}_i - \bar{\epsilon})' \\ &\quad + \frac{1}{n_2} \sum_{i=1}^{p} q_i(\bar{\epsilon}_i - \bar{\epsilon})(\bar{\epsilon}_i - \bar{\epsilon})'\end{aligned}$$

and

$$\frac{1}{n_2} B = \frac{1}{n_2} \sum_{i=1}^{p} \sum_{j=1}^{q_i} (\epsilon_{ij} - \bar{\epsilon}_i)(\epsilon_{ij} - \bar{\epsilon}_i)'$$

since $\mathbf{y}_{ij} = \mu_i + \epsilon_{ij}$, $\bar{\mathbf{y}}_i = \mu_i + \bar{\epsilon}_i$, and $\bar{\mathbf{y}} = \bar{\mu} + \bar{\epsilon}$ where $\bar{\epsilon}_i = \frac{1}{q_i} \sum_{j=1}^{q_i} \epsilon_{ij}$ and $\bar{\epsilon} = \frac{1}{n} \sum_{i=1}^{p} q_i \bar{\epsilon}_i$. Note that ϵ_{ij}'s are independently distributed random vectors with mean $\mathbf{0}$ and covariance matrix I_m. Put

$$E(n) = \frac{1}{\sqrt{n_2}} \sum_{i=1}^{p} q_i[\,(\bar{\epsilon}_i - \bar{\epsilon})(\mu_i - \bar{\mu})' + (\mu_i - \bar{\mu})(\bar{\epsilon}_i - \bar{\epsilon})'\,],$$

and

$$F(n) = \sum_{i=1}^{p} q_i(\bar{\epsilon}_i - \bar{\epsilon})(\bar{\epsilon}_i - \bar{\epsilon})'.$$

Denote μ_i and $\bar{\mu}$ by

$$\mu_i = (\,\mu_{i1}, \ldots, \mu_{im}\,)' \text{ and } \bar{\mu} = (\,\bar{\mu}_{.1}, \ldots, \bar{\mu}_{.m}\,)',$$

where $\bar{\mu}_{.j} = \frac{1}{n} \sum_{i=1}^{p} q_i \mu_{ij}, j = 1, \ldots, m$. Denote ϵ_{ij}, $\bar{\epsilon}_i$, and $\bar{\epsilon}$ by

$$\epsilon_{ij} = (\,\epsilon_{ij1}, \ldots, \epsilon_{ijm}\,)',\ \bar{\epsilon}_i = (\,\bar{\epsilon}_{i.1}, \ldots, \bar{\epsilon}_{i.m}\,)', \text{ and } \bar{\epsilon} = (\,\bar{\epsilon}_{..1}, \ldots, \bar{\epsilon}_{..m}\,)',$$

where $\bar{\epsilon}_{i.r} = \frac{1}{q_i} \sum_{j=1}^{q_i} \epsilon_{ijr}$ and $\bar{\epsilon}_{..r} = \frac{1}{n} \sum_{i=1}^{p} \sum_{j=1}^{q_i} \epsilon_{ijr}, r = 1, \ldots, m$. Then by the construction of Ω

$$\Omega = \sum_{i=1}^{p} q_i(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})'$$

$$= \begin{bmatrix} \Omega_{11} & \cdots & \Omega_{1m} \\ & \vdots & \\ \Omega_{m1} & \cdots & \Omega_{mm} \end{bmatrix},$$

where

$$\Omega_{rl} = \sum_{i=1}^{p} q_i(\mu_{ir} - \bar{\mu}_{.r})(\mu_{il} - \bar{\mu}_{.l}), \text{ for } r, l = 1, \ldots, m.$$

Recall that Ω is a diagonal matrix under H_k, i.e., $\Omega = diag\{n_2\theta_1, \ldots, n_2\theta_k, 0, \ldots, 0\}$. Hence $\mathbf{E}[\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.f}(\mu_{ig} - \bar{\mu}_{.g})] = 0$, and $Var[\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.f}(\mu_{ig} - \bar{\mu}_{.g})] = 0$ for $f = 1, \ldots, m$, $g = k+1, \ldots, m$, so

$$\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.f}(\mu_{ig} - \bar{\mu}_{.g}) = 0 \text{ w.p. 1 for } f = 1, \ldots, m, \ g = k+1, \ldots, m.$$

Let $E_{ij}(n)$ and $F_{ij}(n)$ be the ij^{th} elements of matrices $E(n)$ and $F(n)$, respectively. Then we have

$$E(n) = \frac{1}{\sqrt{n_2}} \sum_{i=1}^{p} q_i[\ \bar{\epsilon}_i(\mu_i - \bar{\mu})' + (\mu_i - \bar{\mu})\bar{\epsilon}_i'\ ]$$

$$= \begin{bmatrix} E_{11}(n) & \cdots & E_{1k}(n) & G_{1,k+1}(n) \\ & \vdots & & \\ E_{k1}(n) & \cdots & E_{kk}(n) & G_{k,k+1}(n) \\ G_{1,k+1}(n)' & \cdots & G_{k,k+1}(n)' & 0 \end{bmatrix},$$

where

$$E_{rr}(n) = \frac{2}{\sqrt{n_2}} \sum_{i=1}^{p} q_i\bar{\epsilon}_{i.r}(\mu_{ir} - \bar{\mu}_{.r}), \text{ for } r = 1, \ldots, k,$$

$$E_{rl}(n) = \frac{1}{\sqrt{n_2}} \sum_{i=1}^{p} q_i[\bar{\epsilon}_{i.r}(\mu_{ir} - \bar{\mu}_{.r}) + \bar{\epsilon}_{i.l}(\mu_{il} - \bar{\mu}_{.l})], \text{ for } r, l = 1, \ldots, k, \ (r \neq l),$$

and for $r = 1, \ldots, k$,

$$G_{r,k+1}(n) = (E_{r,k+1}(n), \ldots, E_{rm}(n))$$

$$= (\frac{1}{\sqrt{n_2}}\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.k+1}(\mu_{ir}-\bar{\mu}_{.r}),\ldots,\frac{1}{\sqrt{n_2}}\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.m}(\mu_{ir}-\bar{\mu}_{.r})).$$

Note that

$$F(n) = \sum_{i=1}^{p} q_i(\bar{\epsilon}_i - \bar{\epsilon})(\bar{\epsilon}_i - \bar{\epsilon})'$$
$$= \begin{bmatrix} F_{11}(n) & \cdots & F_{1k}(n) & H_{1,k+1}(n) \\ & \vdots & & \\ F_{k1}(n) & \cdots & F_{kk}(n) & H_{k,k+1}(n) \\ H_{1,k+1}(n)' & \cdots & H_{k,k+1}(n)' & H_{k+1,k+1}(n) \end{bmatrix},$$

where

$$F_{rl} = \sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.r} - \bar{\epsilon}_{..r})(\bar{\epsilon}_{i.l} - \bar{\epsilon}_{..l}), \text{ for } r,l = 1,\ldots,k,$$

$$H_{r,k+1}(n) = (F_{r,k+1}(n),\ldots,F_{rm}(n))$$
$$= (\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.r} - \bar{\epsilon}_{..r})(\bar{\epsilon}_{i.k+1} - \bar{\epsilon}_{..k+1}),\ldots,\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.r} - \bar{\epsilon}_{..r})(\bar{\epsilon}_{i.m} - \bar{\epsilon}_{..m})),$$

and

$$H_{k+1,k+1}(n) = \begin{bmatrix} F_{k+1,k+1}(n) & \cdots & F_{k+1,m}(n) \\ & & \vdots \\ F_{m,k+1}(n) & \cdots & F_{mm}(n) \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.k+1} - \bar{\epsilon}_{..k+1})^2 & \cdots & \sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.k+1} - \bar{\epsilon}_{..k+1})(\bar{\epsilon}_{i.m} - \bar{\epsilon}_{..m}) \\ & \vdots & \\ \sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.m} - \bar{\epsilon}_{..m})(\bar{\epsilon}_{i.k+1} - \bar{\epsilon}_{..k+1}) & \cdots & \sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.m} - \bar{\epsilon}_{..m})^2 \end{bmatrix}.$$

We will here apply Theorem 1. Note that

$$A(n) = \frac{1}{n_2}A = \Theta + \frac{1}{\sqrt{n_2}}E(n) + \frac{1}{n_2}F(n),$$

$$A_{rr}(n) = \theta_r + \frac{1}{\sqrt{n_2}} E_{rr}(n) + O_p(n_2^{-1}), \text{ for } r = 1, \ldots, k,$$

$$A_{rl}(n) = \frac{1}{\sqrt{n_2}} E_{rl}(n) + O_p(n_2^{-1}), \text{ for } r, l = 1, \ldots, k \ (r \neq l),$$

$$A_{r,k+1}(n) = \frac{1}{\sqrt{n_2}} G_{r,k+1}(n) + O_p(n_2^{-1}), \text{ for } r = 1, \ldots, k,$$

$$A_{k+1,k+1}(n) = \frac{1}{n_2} H_{k+1,k+1}(n),$$

and

$$B(n) = \frac{1}{n_2} B.$$

Denote $U(n)$ and $V(n)$ by

$$U(n) = \sqrt{n_2}\,[\, A(n) - \Theta \,]$$
$$= \sqrt{n_2} \begin{bmatrix} A_{11}(n) - \theta_1 & A_{12}(n) & \cdots & A_{1k}(n) & A_{1,k+1}(n) \\ A_{21}(n) & A_{22}(n) - \theta_2 & \cdots & A_{2k}(n) & A_{2,k+1}(n) \\ & & \vdots & & \\ A_{k1}(n) & A_{k2}(n) & \cdots & A_{kk}(n) - \theta_k & A_{k,k+1}(n) \\ A_{k+1,1}(n) & A_{k+1,2}(n) & \cdots & A_{k+1,k}(n) & \sqrt{n_2} A_{k+1,k+1}(n) \end{bmatrix}$$

and

$$V(n) = \sqrt{n_2}\,[\, B(n) - I_m \,].$$

We should check that $U(n)$ and $V(n)$ have the joint asymptotic distribution in order to use Theorem 1. We can easily show by the Multivariate Central Limit Theorem that

$$[\, U(n),\ V(n) \,] \xrightarrow{L} [\, U,\ V \,].$$

We denote by U_{rr} and V_{rr} the rr^{th} elements of matrices U and V for $r = 1, \ldots, k$, respectively. So by Theorem 2 we have $\sqrt{n_2}(f_r - \theta_r) \xrightarrow{L} U_{rr} - \theta_r V_{rr}$, and the asymptotic distribution of $n_2(f_{k+1}, \ldots, f_m)$ is the same as the distribution of the roots of a $(m-k)\times(m-k)$ matrix $H_{k+1,k+1}-\sum_{r=1}^{k}\frac{1}{\theta_i}G'_{r,k+1}G_{r,k+1}$. By the Central Limit Theorem, we have

$$E_{rr}(n)\xrightarrow{L}U_{rr}\sim N(0,4\theta_r),$$

and

$$V_{rr}(n)\xrightarrow{L}V_{rr}\sim N(0,4\sum_{i=1}^{p}k_i[\mathbb{E}(\epsilon_{i1r}^4)-1]),$$

for $r=1,\ldots,k$. Now we will compute the asymptotic covariance between $E_{rr}(n)$ and $V_{rr}(n)$.

$$\begin{aligned}
Cov(E_{rr}(n),V_{rr}(n)) &= Cov(\frac{2}{\sqrt{n_2}}\sum_{i=1}^{p}q_i\,\bar{\epsilon}_{i.r}\,(\mu_{ir}-\bar{\mu}_{.r}),\frac{1}{\sqrt{n_2}}\sum_{i=1}^{p}\sum_{j=1}^{q_i}(\epsilon_{ijr}^2-1))\\
&= \frac{2}{n_2}\sum_{i=1}^{p}q_i(\mu_{ir}-\bar{\mu}_{.r})\mathbb{E}(\epsilon_{i1r}^3)\\
&\to 2\sum_{i=1}^{p}k_i(\mu_{ir}-\bar{\mu}_{.r})\mathbb{E}(\epsilon_{i1r}^3),
\end{aligned}$$

as $n_2\to\infty$. Now we will see whether or not $\sqrt{n_2}(f_r-\theta_r)$ and $\sqrt{n_2}(f_l-\theta_l)$ are asymptotically independent. To see this we will compute

$$\begin{aligned}
&Cov(E_{rr}(n)-\theta_rV_{rr}(n),E_{ll}(n)-\theta_lV_{ll}(n))\\
&\quad= Cov(E_{rr}(n),E_{ll}(n))-\theta_rCov(V_{rr}(n),E_{ll}(n))\\
&\qquad-\theta_lCov(E_{rr}(n),V_{ll}(n))+\theta_r\theta_lCov(V_{rr}(n),V_{ll}(n)).
\end{aligned}$$

We will compute $Cov(E_{rr}(n),E_{ll}(n))$ for $r,l=1,\ldots,k$, $(r\neq l)$ as follows:

$$\begin{aligned}
&Cov(E_{rr}(n),E_{ll}(n))\\
&\quad= Cov(\frac{2}{\sqrt{n_2}}\sum_{i=1}^{p}q_i\bar{\epsilon}_{i.r}(\mu_{ir}-\bar{\mu}_{.r}),\frac{2}{\sqrt{n_2}}\sum_{i=1}^{p}q_i\bar{\epsilon}_{i.l}(\mu_{il}-\bar{\mu}_{.l}))\\
&\quad= \frac{4}{n_2}\mathbb{E}\left[\left\{\sum_{i=1}^{p}q_i\bar{\epsilon}_{i.r}(\mu_{ir}-\bar{\mu}_{.r})\right\}\left\{\sum_{i=1}^{p}q_i\bar{\epsilon}_{i.l}(\mu_{il}-\bar{\mu}_{.l})\right\}\right]\\
&\quad= \frac{4}{n_2}\sum_{i=1}^{p}\sum_{j=1}^{q_i}(\mu_{ir}-\bar{\mu}_{.r})(\mu_{il}-\bar{\mu}_{.l})\,\mathbb{E}(\epsilon_{ijr}\epsilon_{ijl})\\
&\quad= 0
\end{aligned}$$

since ϵ_{ij} has mean $\mathbf{0}$ and covariance matrix I_m for $i = 1, \ldots, p$ and $j = 1, \ldots, q_i$. We can show similarly by tedious but direct computation that when $1 \leq r, l \leq k$,

$$Cov(V_{rr}(n), E_{ll}(n)) \to 2\sum_{i=1}^{p} k_i(\mu_{il} - \bar{\mu}_{.l})\mathbb{E}(\epsilon_{i1r}^2\epsilon_{i1l}), \tag{8}$$

$$Cov(V_{ll}(n), E_{rr}(n)) \to 2\sum_{i=1}^{p} k_i(\mu_{ir} - \bar{\mu}_{.r})\mathbb{E}(\epsilon_{i1l}^2\epsilon_{i1r}), \tag{9}$$

$$Cov(V_{rr}(n), V_{ll}(n)) \to \sum_{i=1}^{p} k_i(\mathbb{E}(\epsilon_{i1r}^2\epsilon_{i1l}^2) - 1). \tag{10}$$

From (8), (9), and (10) we see that $\sqrt{n_2}(f_r - \theta_r)$ and $\sqrt{n_2}(f_l - \theta_l)$ are generally not asymptotically independent. Let $\mathbf{h}(n) = (\sqrt{n_2}(f_1 - \theta_1), \ldots, \sqrt{n_2}(f_k - \theta_k))$. Here we will show that $\mathbf{h}(n)$ and $n_2(f_{k+1}, \ldots, f_m)$ are asymptotically dependent. Put

$$\mathbf{z}_1 = (\sqrt{q_i}\bar{\epsilon}_{i.1}, \frac{1}{\sqrt{q_i}}\sum_{j=1}^{q_i}(\epsilon_{ij1}^2 - 1); \ldots; \sqrt{q_i}\bar{\epsilon}_{i.k}, \frac{1}{\sqrt{q_i}}\sum_{j=1}^{q_i}(\epsilon_{ijk}^2 - 1))'$$

$$\mathbf{z}_2 = (\sqrt{q_i}\bar{\epsilon}_{i.k+1}, \ldots, \sqrt{q_i}\bar{\epsilon}_{i.m})'$$

and define $\mathbf{z} = (\mathbf{z}_1', \mathbf{z}_2')'$. We will compute some asymptotic covariances in the following way: for $s \neq t$

$$\begin{aligned} Cov(\sqrt{q_i}\bar{\epsilon}_{i.s}, \sqrt{q_i}\bar{\epsilon}_{i.t}) &= \mathbb{E}[(\sqrt{q_i}\bar{\epsilon}_{i.s})(\sqrt{q_i}\bar{\epsilon}_{i.t})] \\ &\to 0 \end{aligned}$$

since $\sqrt{q_i}\bar{\epsilon}_{i.s}$ and $\sqrt{q_i}\bar{\epsilon}_{i.t}$ are asymptotically independent.

$$\begin{aligned} Cov(\frac{1}{\sqrt{q_i}}\sum_{j=1}^{q_i}(\epsilon_{ijs}^2 - 1), \sqrt{q_i}\bar{\epsilon}_{i.t}) &= \mathbb{E}[(\frac{1}{\sqrt{q_i}}\sum_{j=1}^{q_i}(\epsilon_{ijs}^2 - 1))(\frac{1}{\sqrt{q_i}}\sum_{j=1}^{q_i}\epsilon_{ijt})] \\ &= \mathbb{E}[\epsilon_{i1s}^2\epsilon_{i1t}] \end{aligned}$$

which is not zero. These covariance expressions show that, in general, the elements of $\mathbf{z}_1$ are not asymptotically independent of the elements of $\mathbf{z}_2$. Since $\mathbf{h}(n)$ and $n_2(f_{k+1}, \ldots, f_m)$ are functions of $\mathbf{z}_1$ and $\mathbf{z}_2$, respectively, we show that $\mathbf{h}(n)$ and $n_2(f_{k+1},\ldots,f_m)$. are asymptotically dependent. Now we will find the asymptotic distribution of $n_2(f_{k+1},\ldots,f_m)$. Let for $f = k+1,\ldots,m$, as $n \to \infty$,

$$\bar{\epsilon}^{(f)} = \begin{pmatrix} \sqrt{q_1}\bar{\epsilon}_{1.f} \\ \vdots \\ \sqrt{q_p}\bar{\epsilon}_{p.f} \end{pmatrix} \xrightarrow{L} N(\mathbf{0}, I_p), \tag{11}$$

and

$$\bar{\epsilon}_{..f} = \frac{1}{n}\sum_{i=1}^{p} q_i\bar{\epsilon}_{i.f} = \frac{1}{\sqrt{n}}V_p'\bar{\epsilon}^{(f)},$$

where

$$V_p = \frac{1}{\sqrt{n}}\left(\sqrt{q_1}\cdots\sqrt{q_p}\right)' \text{ so that } \|V_p\| = 1.$$

Let

$$V_h = \frac{1}{\sqrt{n_2\theta_h}}\begin{pmatrix} \sqrt{q_1}(\mu_{1h} - \bar{\mu}_{.h}) \\ \vdots \\ \sqrt{q_p}(\mu_{ph} - \bar{\mu}_{.h}) \end{pmatrix}.$$

For $h = 1,\ldots,k$ then

$$V_h'V_p = \frac{1}{\sqrt{nn_2\theta_h}}\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h}) = 0$$

$$V_h'V_h = \frac{1}{n_2\theta_h}\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})^2 = 1$$

and for $h, h' = 1,\ldots,k$ $(h \neq h')$ then

$$V_h'V_{h'} = \frac{1}{n_2\sqrt{\theta_h\theta_{h'}}}\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\mu_{ih'} - \bar{\mu}_{.h'}) = 0.$$

If we choose a set of $V_{k+1},\ldots,V_{p-1}$ so that $V_j'V_j = 1$ and $V_i'V_j = 0$ for $i,j = 1,\ldots,p$, then a basis $\{V_1,\ldots,V_p\}$ is orthonormal. Hence the asymptotic distribution of $\bar{\epsilon}^{(f)}$ is the same as the distribution of $\sum_{i=1}^{p} V_i Z_{if}$, where $Z_{1f}, \ldots, Z_{pf}$ are i.i.d. $N(0,1)$ random variables for $f = k+1, \ldots, m$. We can easily check that

$$\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})^2 = \|\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p\|^2. \tag{12}$$

Since $V_p'(\sum_{i=1}^{p} V_i Z_{if}) = Z_{pf}$ and so $\| \sum_{i=1}^{p-1} V_i Z_{if}\|^2 = \sum_{i=1}^{p-1} Z_{if}^2$, the asymptotic distribution of $\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})^2$ is the same as the distribution of $\sum_{i=1}^{p-1} Z_{if}^2$ which is χ_{p-1}^2. And we can also check that

$$\begin{aligned}&\frac{1}{n_2}\sum_{h=1}^{k} \frac{1}{\theta_h}[\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})][\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})] \\ &= \sum_{h=1}^{k} \left[V_h'[\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p]\right]^2 = \sum_{h=1}^{k} [V_h'\bar{\epsilon}^{(f)}]^2. \end{aligned} \tag{13}$$

Hence the asymptotic distribution of $\sum_{h=1}^{k} [V_h'\bar{\epsilon}^{(f)}]^2$ is the same as the distribution of $\sum_{h=1}^{k} Z_{hf}^2$. Since $\sum_{i=1}^{p-1} Z_{if}^2 - \sum_{i=1}^{k} Z_{if}^2 = \sum_{i=k+1}^{p-1} Z_{if}^2$, the asymptotic distribution of

$$\begin{aligned}&\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})^2 \\ &-\frac{1}{n_2}\sum_{h=1}^{k} \frac{1}{\theta_h}[\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})][\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})] \end{aligned} \tag{14}$$

is the same as the distribution of $\sum_{i=k+1}^{p-1} Z_{if}^2$ which is χ_{p-1-k}^2. For $f, g = k+1, \ldots, m$, the asymptotic distribution of

$$\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})(\bar{\epsilon}_{i.g} - \bar{\epsilon}_{..g}) = (\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p)'(\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p)$$

is the same as the distribution of $\sum_{i=1}^{p-1} Z_{if}Z_{ig}$. We can check easily that

$$\sum_{h=1}^{k} \frac{1}{n_2\theta_h}[\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})][\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.g} - \bar{\epsilon}_{..g})]$$

$$= \sum_{h=1}^{k} [V_h'[\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p]]\,[V_h'[\bar{\epsilon}^{(g)} - (V_p'\bar{\epsilon}^{(g)})V_p]].$$

The asymptotic distribution of

$$\sum_{h=1}^{k} [V_h'[\bar{\epsilon}^{(f)} - (V_p'\bar{\epsilon}^{(f)})V_p]]\,[V_h'[\bar{\epsilon}^{(g)} - (V_p'\bar{\epsilon}^{(g)})V_p]]$$

is the same as the distribution of $\sum_{h=1}^{k} Z_{hf}Z_{hg}$. Hence the asymptotic distribution of

$$\sum_{i=1}^{p} q_i(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})(\bar{\epsilon}_{i.g} - \bar{\epsilon}_{..g})$$
$$- \sum_{h=1}^{k} \frac{1}{n_2\theta_h}[\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.f} - \bar{\epsilon}_{..f})][\sum_{i=1}^{p} q_i(\mu_{ih} - \bar{\mu}_{.h})(\bar{\epsilon}_{i.g} - \bar{\epsilon}_{..g})]$$

is the same as the distribution of $\sum_{i=k+1}^{p-1} Z_{if}Z_{ig}$ for $f, g = k+1, \ldots, m$. So, by the definition of Wishart distribution, the asymptotic distribution of

$$H_{k+1,k+1} - \sum_{r=1}^{k} \frac{1}{\theta_h} G'_{r,k+1}G_{r,k+1}$$

is $W_{m-k}(n_1 - k, I_{m-k})$. So the asymptotic distribution of $n_2(f_{k+1}, \ldots, f_m)$ is the distribution of the roots of $(m-k) \times (m-k)$ matrix having the $W_{m-k}(n_1 - k, I_{m-k})$ distribution. ■

We note that $\sqrt{n_2}(f_1 - \theta_1), \ldots, \sqrt{n_2}(f_k - \theta_k)$ are usually asymptotically dependent but asymptotically independent for normal populations. We also notice that the asymptotic distribution of $(n_2 f_{k+1}, \ldots, n_2 f_m)$ for non-normal populations is the same as the asymptotic distribution of $(n_2 f_{k+1}, \ldots, n_2 f_m)$ for normal populations.

REFERENCES

(1) Amemiya, Y. (1986). Limiting distributions of certain characteristic roots under general conditions and their applications. *Technical Report No. 17, Econometric Workshop, Stanford University.*

(2) Amemiya, Y. (1990). A note on the limiting distribution of certain characteristic roots. *Statistics and Probability Letters* **9** 465-470.

(3) Anderson, T.W. (1951). The asymptotic distribution of certain characteristic roots and vectors, in: J. Neyman, ed., *Proceeding 2nd Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley, CA) 122-148.

(4) Anderson, T.W. (1963). The asymptotic theory for principal component analysis. *Annals of Mathematical Statistics* **34** 122-148.

(5) Anderson, T.W. (1989). The asymptotic distribution of characteristic roots and vectors in multivariate components of variance, in : L.J. Gleser et al., eds., *Contributions to Probability and Statistics: Essays in honor of Ingram Olkin* (Springer, New York) 177-196.

(6) Bai, Z.D. (1984). A note on asymptotic joint distribution of the eigenvalues of a noncentral multivariate F matrix. *Technical Report No. 84-49, Center for Multivariate Anal., University of Pittsburgh* .

(7) Davis, A.W. (1977). Asymptotic theory for principal component analysis: Nonnormal case. *Australian Journal of Statistics* **19** 206-212.

(8) Fang, C. and Krishnaiah, P.R. (1982). Asymptotic distributions of functions of the eigenvalues of some matrices for non-normal populations. *Journal of Multivariate Analysis* **12** 39-63.

(9) Fujikoshi, Y. (1980). Asymptotic expansion for the distributions of the sample roots under nonnormality. *Biometrika* **67** 45-51.

(10) Hsu, P.L. (1941). On the limiting distribution of roots of a determinantal equation. *Journal of London Mathematical Society* **16** 183-194.

(11) Hwang, C. (1991). *Model selection methods in discriminant analysis.* Unpublished Ph.D Thesis, University of Michigan, Ann Arbor, Michigan.

(12) Muirhead, R.J. (1982). Aspects of Multivariate Statistical Theory. *Wiley, New York.*

(13) Waternaux, C.M. (1976). Asymptotic distribution of the sample roots for a nonnormal population. *Biometrika* **63** 639-645.