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Asymptotics of a Class of Markov Processes Generated by $X_{n+1} = f(X_n) + \varepsilon_{n+1}$†

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ABSTRACT

We consider the Markov process $\{X_n\}$ on R which is generated by $X_{n+1} = f(X_n) + \varepsilon_{n+1}$. Sufficient conditions for irreducibility and geometric ergodicity are obtained for such Markov processes. In additions, when $\{X_n\}$ is geometrically ergodic, the functional central limit theorem is proved for every bounded functions on R.

KEYWORDS: Markov chain, Irreducibility, Ergodicity, Geometric ergodicity, Functional central limit theorem.

1. PRELIMINARIES

Suppose $\{X_n\}$ is a Markov process taking values in some arbitrary space (S, ζ) with n-step transition probability

$$P^{(n)}(x, B) = \Pr(X_n \in B \mid X_0 = x), \quad x \in S, B \in \zeta.$$

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We shall call a Markov process with transition probabilities $P^{(n)}(x,B)$ φ-irreducible for some non-trivial σ - finite measure φ on ζ if whenever $\varphi(B)>0$,

$$\sum_{n=1}^{\infty} 2^{-n} P^{(n)}(x,B) > 0 \qquad \text{for every } x \in S.$$

A non-trivial σ-finite measure π on ζ is called subinvariant for $\{X_n\}$ if

$$\int P(x,B)\pi(dx) \le \pi(B), \quad B \in \zeta. \tag{1.1}$$

π is called invariant if equality holds in (1.1) for all $B \in \zeta$.

If $\{X_n\}$ is a φ-irreducible process, there is a subinvariant measure stronger than φ (see [Jain and Jamison (1967)]), and a subinvariant measure which is finite is necessarily invariant. If the unique invariant measure π is finite, then we shall call $\{X_n\}$ positive recurrent.

We call $\{X_n\}$ geometrically ergodic if it is positive recurrent and there exists positive $\rho < 1$ such that $\| P^{(n)}(x,\cdot) - \pi(\cdot) \| = O(\rho^n)(n \to \infty)$ for π - a.s. $x \in S$, where $\|\cdot\|$ denotes the total variation norm and O stands for the usual "big O".

When using a Markov process as a model, it is often of great importance to know whether the model is positive recurrent, or whether the model is geometrically ergodic. There are extensive literature on these subjects for the case that $\{X_n\}$ is irreducible (see [Jain and Jamison(1967)], [Lee(1988)], [Lee(1991)], [Tong(1990)], [Tweedie(1975)] etc.). For the case that $\{X_n\}$ is non-irreducible, see [Bhattacharya and Lee (1988)].

Let $\{X_n\}$ be a φ-irreducible Markov process on (S,ζ) with transition probabilities $P^{(n)}(x,\cdot)$. Call a set $B \in \zeta$ small if $\varphi(B) > 0$ and for every $A \in \zeta$ with $\varphi(A) > 0$, there exists j such that

$$\inf_{x \in B} \sum_{n=1}^{j} P^{(n)}(x,A) > 0.$$

For an irreducible, aperiodic Markov process $\{X_n\}$ with state space $(S,\zeta), \zeta$ is countably generated, following theorem has proved by Nummelin and Touminen (1982).

Theorem 1.1. Assume that there exist a nonnegative measurable function g on S, a small set $B \in \zeta$, and real numbers $r > 1, \varepsilon > 0$ such that

$$\int P(x,dy)g(y) \le (1/r)g(x) - \varepsilon, \quad x \in B^c,$$

$$\sup_{x\in B} \int_{B^C} P(x,dy)g(y) < \infty.$$

Then $\{X_n\}$ is geometrically ergodic.

In this paper, we are interested in the process of $\{X_n\}$ which is generated by the stochastic difference equation of the form

$$X_{n+1} = f(X_n) + \varepsilon_{n+1}, \quad n \ge 0, \tag{1.2}$$

where f is a measurable function on R into R , $\{\varepsilon_n : n \ge 1\}$ is sequence of independent, indentically distributed random variables on R with distribution Q, and X_0 is arbitrary but independent of ε_n.

Let $\mathcal{B}$ be the class of Borel sets of R and μ the Lebesgue measure.

Then $\{X_n : n \ge 0\}$ with n-step transition probability function

$$P^{(n)}(x,B) = \Pr(X_n \in B \mid X_0 = x), \quad x \in R, \quad B \in \mathcal{B},$$

forms a Markov process with state space (R, $\mathcal{B}$, μ).

In section 2, we give sufficient conditions for irreducibility and geometric ergodicity. In section 3, we find a class of functions h in $L^2(R,\pi)$, for which functional central limit theorem holds.

2. IRREDUCIBILITY AND GEOMETRIC ERGODICITY

Lemma 2.1. For $\{X_n\}$ in (1.2), if f is continuous, then for sequence x_n in

R converging to x, $P(x_n, \cdot)$ converges weakly to $P(x, \cdot)$ as $n \to \infty$.

Proof. Suppose that g is a real-valued bounded continuous function on R and that x_n converges to x as $n \to \infty$. Then

$$\begin{aligned}\int g(z)P(x_n, dz) &= \int g(z + f(x_n))Q(dz) \\ &\to \int g(z + f(x))Q(dz), \text{by bounded convergence theorem} \\ &= \int g(z)P(x, dz).\end{aligned}$$

Throughout this paper, we assume that f in equation (1.2) is continuous.

Theorem 2.2. If Q has a nonzero absolutely continuous component with respect to μ with density q which is positive a.e. $[\mu]$ on R, then the Markov process $\{X_n\}$ is aperiodic and μ-irreducible.

Proof. $P(x, B) = Q(B - f(x)) = \int_{B-f(x)} q(y)dy = 0 \Leftrightarrow \mu(B - f(x)) = 0 \Leftrightarrow \mu(B) = 0$
because of the translation invariance of μ.

The following theorem which was proved by O. Lee(1988) has weakened the condition on Q for $\{X_n\}$ to be irreducible.

Theorem 2.3. Suppose Q has a nonzero absolutely continuous component with respect to μ whose density q is positive on a nonempty open set V. Define

$$V_x^{(1)} = f(x) + V, \qquad V_x^{(n+1)} = \{f(z) + V : z \in V_x^{(n)}\}$$

$$V_x = \bigcup_{n=1}^{\infty} V_x^{(n)}, \qquad W = \bigcap_{x \in R} V_x$$

If there exists $A \in \mathcal{B}$ with $A \subset W$ and $\mu(A) > 0$, then the process is $\varphi-$ irreducible where $\varphi(B) = \mu(B \cap A)$ for Borel set B.

Let

$$\underline{\alpha} = \varliminf_{x \to -\infty} f(x)/x, \overline{\alpha} = \varlimsup_{x \to -\infty} f(x)/x, \underline{\beta} = \varliminf_{x \to \infty} f(x)/x, \overline{\beta} = \varlimsup_{x \to \infty} f(x)/x.$$

Make the assumptions on $\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}$, as follows :

Assumption I

(a) $0 < \underline{\alpha} \le \overline{\alpha} < 1, \qquad 0 < \underline{\beta} \le \overline{\beta} < 1;$

(b) $-\infty < \underline{\alpha} \le \overline{\alpha} < 0, \qquad 0 < \underline{\beta} \le \overline{\beta} < 1;$

(c) $0 < \underline{\alpha} \le \overline{\alpha} < 1, \qquad -\infty < \underline{\beta} \le \overline{\beta} < 0\,;$

(d) $\overline{\alpha} < 0, \qquad \overline{\beta} < 0, \qquad \underline{\alpha\beta} < 1.$

Theorem 2.4. Suppose Q has a density function q on R which is positive everywhere and $E\varepsilon_1 = 0$. Then each of the assumptions I is a sufficient condition for the existence of the unique invariant probability for $\{X_n\}$.

Proof. See [C. Lee (1991)]

The following is proved, by modifying the idea of C. Lee (1991).

Theorem 2.5. Let $\{X_n\}$ be the process obtained by (1.2). If Q satisfies one of the conditions on theorem 2.2 and theorem 2.3, and $E|\varepsilon_1| < \infty$, then each one of the assumptions I is sufficient for the geometric ergodicity of $\{X_n\}$

Proof. Since f is continuous, $x \to \int g(y)P(x,dy)$ is continuous for every real-valued bounded continuous function g. If we set $\mathcal{X}$ as the support of subinvariant measure π, then the assumption ensures that $\mathcal{X}$ is second category. Hence every compact set is small (see [Cogburn (1975)])

If we assume $\quad g(x) = \begin{cases} ax & \text{if } x \ge 0 \\ b|x| & \text{if } x < 0 \end{cases}$ for some $0 < a, b < \infty$,

$$\begin{aligned} \int g(y)P(x,dy) &= \int g(f(x)+z)Q(dz) \\ &= \int_{f(x)+z\ge 0} a(f(x)+z)Q(dz) - \int_{f(x)+z<0)} b(f(x)+z)Q(dz) \\ &\le aE|\varepsilon_1| + bE|\varepsilon_1| + af(x)\int_{f(x)+z\ge 0} Q(dz) - bf(x)\int_{f(x)+z<0} Q(dz). \end{aligned}$$

Hence

if $f(x) \geq 0, \int g(y)P(x,dy) \leq C + af(x)$ and
if $f(x) < 0, \int g(y)P(x,dy) \leq C - bf(x)$, where $C = (a+b)E|\varepsilon_1|$

Moreover, for a compact set B

$$\sup_{x\in B} \int_{B^c} g(y)P(x,dy) \leq C + \max\{a,b\} \sup_{x\in B} |f(x)| < \infty,$$

since f is continuous. To prove the geometric ergodicity of $\{X_n\}$, it remains to show that the existence of nonnegative measurable function g, compact set B, real numbers $r > 1, \varepsilon > 0$ such that

$$\int g(y)P(x,dy) < (1/r)g(x) - \varepsilon, \qquad x \in B^c.$$

Let $\varepsilon > 0$ be arbitrary but fixed.

(a) Suppose $0 < \underline{\alpha} \leq \overline{\alpha} < 1, \quad 0 < \underline{\beta} \leq \overline{\beta} < 1$.

Define $g(x) = |x|, x \in R$. Choose $\theta, \theta', \theta'' > 0$ such that

$$0 < \underline{\alpha} - \theta < \overline{\alpha} + \theta < 1 < \overline{\alpha} + \theta', \quad 0 < \underline{\beta} - \theta < \overline{\beta} + \theta < \overline{\beta} + \theta'' < 1.$$

By definitions of $\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}$, there exist $M_{11}, M_{12} > 0$ such that
for $x < -M_{11}$, $\quad (\underline{\alpha} - \theta)x \geq f(x) \geq (\overline{\alpha} + \theta)x$ and
for $x > M_{12}$, $\quad (\underline{\beta} - \theta) \leq f(x) \leq (\overline{\beta} + \theta)x$.
Now let

$$r_1 = (\overline{\alpha} + \theta')/(\overline{\alpha} + \theta), \quad r_2 = (\overline{\beta} + \theta'')/(\overline{\beta} + \theta).$$

For $x < -M_{11}$,

$$\begin{aligned} \int g(y)P(x,dy) &\leq C - f(x) \\ &\leq C + (\overline{\alpha} + \theta)(-x) \\ &\leq (1/r_1)((\overline{\alpha} + \theta')(-x) + r_1 C) \end{aligned}$$

Since $(\overline{\alpha} + \theta') > 1$, there exist $M'_{11} > M_{11}$ such that if $x < -M'_{11}$,

$$(\overline{\alpha}+\theta')(-x)+r_1C<(-x)-r_1\varepsilon.$$

Therefore if $x<-M'_{11}$,

$$\begin{aligned}\int g(y)P(x,dy) &< (1/r_1)(-x)-\varepsilon \\ &= (1/r_1)g(x)-\varepsilon.\end{aligned}$$

On the other hand if $x>M_{12}$

$$\begin{aligned}\int g(y)P(x,dy) &\leq C+(\overline{\beta}+\theta)x \\ &\leq (1/r_2)((\overline{\beta}+\theta'')x+r_2C).\end{aligned}$$

We may choose $M'_{12}>M_{12}$ such that for $x>M'_{12}$,

$$(\beta+\theta'')x+r_2C<x-r_2\varepsilon.$$

Hence if $x>M'_{12}, \int g(y)P(x,dy)\leq(1/r_2)g(x)-\varepsilon$.

(b) Suppose $-\infty<\underline{\alpha}\leq\alpha<0, \quad 0<\underline{\beta}\leq\beta<1$.
Suppose $-b<\underline{\alpha}$ for some b, $\quad 0<b<\infty$.

Define $$g(x)=\begin{cases} x & \text{if } x\geq 0 \\ b|x| & \text{if } x<0.\end{cases}$$

Choose $r_3>1$ such that $-(1/r_3)b<\underline{\alpha}$.
We may take $\theta,\theta'',\theta_1>0$ with $\theta<\theta_1$ such that

$$-b<\underline{\alpha}-\theta<\overline{\alpha}+\theta<0, \quad -(1/r_3)b+\theta_1<\underline{\alpha},$$

$$0<\underline{\beta}-\theta<\overline{\beta}+\theta<\overline{\beta}+\theta''<1.$$

Now choose M_{21} such that if $x<-M_{21}$, $(\underline{\alpha}-\theta)x\geq f(x)\geq(\overline{\alpha}+\theta)\,x$.
For $x<-M_{21}, f(x)>0$ implies

$$\begin{aligned}\int g(y)P(x,dy) &\leq C+(\underline{\alpha}-\theta)\,x \\ &\leq (1/r_3)b(-x)+(\theta-\theta_1)(-x)+C.\end{aligned}$$

Choose $M'_{21} > M_{21}$ such that if $x < -M'_{21}$, $(\theta - \theta_1)(-x) + C \le -\varepsilon$ and therefore we have

$$\int g(y)P(x,dy) < (1/r_3)g(x) - \varepsilon.$$

By the second part of the case (a), we have

$$\int g(y)P(x,dy) < (1/r_2)g(x) - \varepsilon, \quad \text{if } x > M'_{12}.$$

(c) Suppose $0 < \underline{\alpha} \le \alpha < 1, \quad -\infty < \underline{\beta} \le \beta < 0$.
Suppose $-a < \underline{\beta}$ for some a, $0 < a < \infty$.

Define $$g(x) = \begin{cases} ax & \text{if } x \ge 0 \\ |x| & \text{if } x < 0. \end{cases}$$

Choose $r_4 > 1$ such that $-(1/r_4)a < \underline{\beta}$. We pick θ, θ', $\theta_2 > 0$, $\theta < \theta_2$ such that

$$0 < \underline{\alpha} - \theta < \overline{\alpha} + \theta < 1 < \overline{\alpha} + \theta',$$

$$-a < \underline{\beta} - \theta < \overline{\beta} + \theta < 0, \quad -(1/r_4)a + \theta_2 < \underline{\beta}.$$

There exist $M_{32} > 0$ such that for $x > M_{32}$,

$$\int g(y)P(x,dy) \le (1/r_4)ax + (\theta - \theta_2)\, x + C.$$

Since $\theta - \theta_2 < 0$, We may choose $M'_{32} > M_{32}$ such that if $x > M'_{32}$,

$$\int g(y)P(x,dy) \le (1/r_4)g(x) - \varepsilon.$$

By the first part of case (a),

$$\int g(y)P(x,dy) < (1/r_1)g(x) - \varepsilon, \quad \text{if } x < -M'_{11}.$$

(d) Suppose $\overline{\alpha} < 0$, $\overline{\beta} < 0$, $\underline{\alpha\beta} < 1$.
In this case we may choose $b > 0$ such that

$$-b < \underline{\alpha} \le \overline{\alpha} < 0, \qquad -1/b < \underline{\beta} \le \overline{\beta} < 0.$$

Define $$g(x) = \begin{cases} x & \text{if } x \geq 0 \\ b|x| & \text{if } x < 0. \end{cases}$$

By case (b) if $x < -M'_{21}$,

$$\int g(y)P(x,dy) < (1/r_3)g(x) - \varepsilon.$$

On the other hand, choose $r_5 > 1$ such that $-(1/r_5)(1/b) < \underline{\beta}$.
We may have $\theta, \theta_3 > 0$, with $\theta < \theta_3$ which satisfy

$$-1/b < \underline{\beta} - \theta < \overline{\beta} + \theta < 0, \quad -(1/r_5)(1/b) + \theta_3 < \underline{\beta}.$$

Now choose $M_{42} > 0$ such that for $x > M_{42}$

$$\begin{aligned} \int g(y)P(x,dy) &\leq C + b(\underline{\beta} - \theta)(-x) \\ &\leq C + b((1/r_5)(1/b) - \theta_3)\,x + \theta x). \end{aligned}$$

There exists $M'_{42} > M_{42}$ such that for $x > M'_{42}$

$$\int g(y)P(x,dy) < (1/r_5)g(x) - \varepsilon.$$

Now let $M = \max\{M'_{11}, M'_{12}, M'_{21}, M'_{32}, M'_{42}\}$. If we take $r = \min\{r_i : 1 \leq i \leq 5\}, B = [-M, M]$, then for each case, we have

$$\int g(y)P(x,dy) < (1/r)g(x) - \varepsilon, \qquad x \in B^c$$

which concludes our proof.

Remark 2.6. Another type of sufficient conditions for geometric ergodicity of $\{X_n\}$ can be found on page 128, Tong(1990).

3. FUNCTIONAL CENTRAL LIMIT THEOREM

In this section, we let $\{X_n\}$ be the Markov process generated by (1.2) which satisfies the assumptions on theorem 2.5 with π as its invariant initial distribution.

It is known that $\{X_n\}$ with $X_0 \sim \pi$ becomes a stationary ergodic Markov process (see [Breiman (1968)]).

Our aim is to obtain the functional central limit theorems for

$$Y_n(t) = n^{-1/2} \sum_{j=0}^{[nt]} (h(X_j) - \int h d\pi), \qquad 0 \le t < \infty \tag{3.1}$$

for a class of functions h in $L^2(R, \pi)$ where [nt] denotes the integer part of nt.

The process defined by (3.1) takes values in the space $D[0, \infty)$ of real-valued right continuous function on $D[0, \infty)$, having left hand limits with the Skorohod topology. The distribution of Y_n is then a probablity measure on the Borel σ-field of $D[0, \infty)$ and its convergence in distribution to a Brownian motion means the weak convergence of this sequence of distributions to a Wiener measure.

The transition operator T on $L^2(R, \pi)$ is defined by

$$(Th)(x) = \int h(y)P(x, dy), \qquad h \in L^2(R, \pi).$$

Then $$(T^n h)(x) = \int h(y)P^{(n)}(x, dy), \qquad h \in L^2(R, \pi).$$

Let I be the identity operator. Write $\overline{h} = \int h d\pi$. $|| \cdot ||$ denotes the L^2-norm in $L^2(R, \pi)$.

Therorem 3.1. If $\{X_n\}$ is geometrically ergodic, there exists positive $\rho < 1$ such that

$$\int \pi(dx) \, || \, P^n(x, \cdot\,) - \pi(\,\cdot\,) \, || = O(\rho^n)(n \to \infty).$$

Proof. See [Nummelin and Touminen (1982)]

Let B(R) be the linear space of all real-valued bounded measurable functions on R.

Theorem 3.2. For every $h \in B(R)$, the process Y_n in (3.1) converges in distribution to a Brownian motion with mean zero and variance parameter

$$\| g \|_2^2 - \| Tg \|_2^2 \qquad \text{where } (T-I)g = h - \bar{h}.$$

Proof. Suppose $h \in B(R)$ with $h(x) \leq B, {}^\forall x \in R, 0 < B < \infty$, and take

$$g = -\sum_{n=0}^{\infty} T^n (h - \bar{h}). \tag{3.2}$$

If we apply T on both side of (3.2), then we have $(T-I)g = h - \bar{h}$.
Moreover,

$$\begin{aligned} \| T^n(h-\bar{h}) \|_2^2 &= \int \Big(\int h(y) P^{(n)}(x,dy) - \int h(y)\pi(dy) \Big)^2 \pi\,(dx) \\ &\leq \int \Big(B \,\| P^{(n)}(x,\cdot) - \pi(\cdot) \| \Big)^2 \pi\,(dx) \end{aligned}$$

Since $\{X_n\}$ is geometrically ergodic, there exist positive $\rho < 1$, measurable function M so that $M < \infty$, π- a.e. and $\| P^n(x,\cdot) - \pi(\cdot) \| \leq M(x)\rho^n$ as $n \to \infty$. Moreover by the theorem 3.1,

$$\int \pi\,(dx) \| P^{(n)}(x,\cdot) - \pi(\cdot) \| \leq O(\rho^n) \qquad \text{as } n \to \infty.$$

Therefore for sufficiently large n, we have

$$\begin{aligned} \| T^n(h-\bar{h}) \|_2^2 &\leq 2B^2 \int \pi\,(dx) \,\| P^{(n)}(x,\cdot) - \pi(\cdot) \| \\ &\leq 2B^2 K \rho^n \qquad \text{for some } 0 < K < \infty, \end{aligned}$$

which implies $\sum_{n=0}^{\infty} \| T^n(h-\bar{h}) \|_2 < \infty$, and hence $h - \bar{h}$ belongs to the range of T-I.

$$\begin{aligned} \sum_{j=0}^{n} (h(X_j) - \bar{h}) &= \sum_{j=0}^{n} (Tg(X_j) - g(X_j)) \\ &= \sum_{j=1}^{n+1} (Tg(X_{j-1}) - g(X_j)) + (g(X_{n+1}) - g(X_0)). \end{aligned}$$

Since $\{Tg(X_{j-1}) - g(X_j) : j \geq 0)\}$ is a stationary ergodic sequence of martingale differences, the functional central limit theorem follows (see [Billingsley(1968)], [Gordin and Lifsic(1978)]). The variance parameter of the limiting Brownian motion is $E(Tg(X_{j-1}) - g(X_j))^2 = \| g \|_2^2 - \| Tg \|_2^2$.

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