Successive Over Relaxation 방법을 이용한 임의 형상 단면의 비틀림 특성연구

Successive Over Relaxation Method in Torsion Problem for General Shaped Section

윤태양* **김태진****

Yoon, Tae Yang Kim, Tae Jin

요 약

본 연구에서는 임의형상의 속이 찬 또는 中孔 단면에 대한 비틀림 상수및 전단응력 계산용 범용 프로그램 TORSION을 소개한다. Membrane Analogy로부터 차분법을 유도한뒤 Successive Over Relaxation 방법을 사용하여 정해를 효과적으로 얻을 수 있었다. 사각단면 이외 단면의 경우, 경계조건을 전체 Boundary Mesh에 대하여 계산 입력해야하는 번거러움을 해결하기 위하여 자동경계 생성기법을 사용하였다. 개발된 프로그램 TORSION을 사용한 Parametric 해석을 통하여 비교적 정확한 일반 中孔 단면용 비틀림상수 계산을 위한 개략식을 제안한후, 콘크리트 표준시방서에 제시된 방법과 비교하였다.

Abstract

In this study a general computer program TORSION is presented to calculate a torsion constant and shear stress for any section shapes with or without holes. From the *membrane analogy* a finite difference method with Successive Over Relaxation are used to accelerate the iteration. Automatic Boundary Generating Technique relieved the cumbersome efforts to prepare the input values of boundary for the general shaped section. Parametric analyses were conducted using the program TORSION. Approximate formulas for the general shaped hollow sections were proposed and compared with those from the Korean Concrete Standard.

Keywords : TORSION, torsion constant, shear stress, membrane analogy, successive over relaxation, automatic boundary generating, general shaped section, finite difference method.

* 정회원, 삼우건설기술연구원 이사, 공학박사

** 정회원, 삼우건설기술연구원 연구원

• 본 논문에 대한 토의를 1994년 4월 30일까지 학회로 보내주시면 1994년 6월호에 토의회답을 게재하겠습니다.

1. Introduction

The torsion problem in various shaped sections have a long history and at the same time have been a continuous research subject. Since the problem of torsion in an elastic circular member was first studied by Coulomb[1,2] in 1784, solutions for those of non-circular cross sections such as rectangular, elliptical, triangular and others were presented to the French Academy by Saint-Venant[1,2,3] in 1853. Prandtl[1,2,3,4] introduced a very valuable *membrane analogy* between the stress function in the torsion problem and the deflection of a membrane under uniform loading. Following the membrane analogy, approximate torsion equations for flanged sections such as T, L and I shape were suggested by Bach[1] in 1911. Solutions of torsion problem using other method such as energy method[2] and hydrodynamic analogies[2] were also introduced. Assuming the shear stress is uniform through the thickness, Bredt[1,2] derived a very simple equation form for a thin tube. Finite difference method in the application to the various torsion problems were published by Southwell and his students[5] in 1946 whereas a general concept of finite element method using different element types to solve torsion problem was explained by Bickford[6]. The difficulty remains however because each method can be applied only to one or some of the cases and sometimes it requires cumbersome efforts.

2. Governing Equations

membrane analogy[1,2,3,4] stems from the observation that both the stress function and the deflection are governed by Laplace's harmonic differential equation, and also must satisfy the same boundary conditions. When a homogeneous membrane such as a soap film subjected to a uniform load per unit area, q, within a boundary having a shape similar to the cross section of a given torsion beam, the uniform tensile force per unit length of the membrane denoted by S produces a deflection w. For an isolated infinitesimal element, the equilibrium of forces in the vertical direction gives the following equation.

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{S} \qquad (1)$$

where w = the deflection of the membrane, q = the uniform load per unit area of the membrane and S = the uniform tensile force per unit length of the membrane. Comparing with the Airy's stress function[6], volume of the deflected membrane equals to the half of the torsional moment and the slope of the membrane matches with the shear stress. At each boundaries, both deflection of the membrane and the Airy's stress function are zeros. Then, the torsion constant K_T is related to the volume v of the soap bubble by the equation

$$K_T = \frac{4S}{q} v \qquad (2)$$

where K_T = the torsion constant and v = the volume bounded by the deflected membrane. The shear stress at any point due to torque T is related to the slope $m = \partial w / \partial n$ and volume v by the equation

$$\frac{\tau}{T} = \frac{m}{2v} \qquad (3)$$

where τ = the shear stress at any point, T = the torque, $m = \partial w / \partial n$ = the slope of the membrane and n indicates the normal direction. It was shown[2] that in the case of section with holes the stress function, w, must not only satisfy Eq. (1) but also the boundary of each hole we must have

$$-\int \frac{\partial w}{\partial n} ds = A\frac{q}{S} \quad (4)$$

where A denotes the area of the hole. Eq. (4) means that the load uniformly distributed over the area of the hole is balanced by the tensile forces in the membrane. In other word, the hole is represented by a weightless absolutely rigid plate which can move perpendicularly to the initial plane of the stretched membrane.

3. Solution Method

Finite-Difference Method was used to solve the differential equation (1). Using the second polynomial with central interpolation we obtain the approximate values of second derivatives such as

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=h_x} \simeq \frac{w_2-2w_1+w_0}{h_x^2}$$
$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=h_y} \simeq \frac{w_2-2w_1+w_0}{h_y^2} \quad (5)$$

where h_x = the x-direction interval and h_y = the y-direction interval.
Substituting Eq. (5) into Eq. (1) gives

$$\alpha w_{i-1,j}-2(1+\alpha)w_{i,j}+\alpha w_{i+1,j}+w_{i,j-1}+w_{i,j+1} = -(h_y)^2\frac{q}{S} \quad (6)$$

where $\alpha = (h_y/h_x)^2$, i = the x-direction node and j = the y-direction node. Assuming an equal interval in x,y directions Eq. (6) can be simplified as

$$w_{i-1,j}+w_{i+1,j}-4w_{i,j}+w_{i,j-1}+w_{i,j+1} = -(h)^2\frac{q}{S} \quad (7)$$

where h denotes the interval of the meshes. Various iterative methods can be used to solve this equation. Southwell[5] use *Relaxation Method* to treat the differential equation as

$$w_{i,j}=w_{i,j}^{(0)}+\frac{1}{4}\left\{(h)^2\frac{q}{S}+w_{i+1,j}^{(0)}+w_{i-1,j}^{(0)} -4w_{i,j}^{(0)}+w_{i,j+1}^{(0)}+w_{i,j-1}^{(0)}\right\} \quad (8)$$

where the superscript (0) indicates previous iteration number. Replacing simultaneous terms calculated at the previous iteration with successive terms evaluated at the present iteration process and assuming q = 4 S we get

$$w_{i,j}^{(p+1)}=w_{i,j}^{(p)}+h^2+\frac{1}{4}\left\{w_{i+1,j}^{(p)}+w_{i-1,j}^{(p+1)} -4w_{i,j}^{(p)}+w_{i,j+1}^{(p)}+w_{i,j-1}^{(p+1)}\right\} \quad (9)$$

where the superscript p indicates the iteration number. If we define the residual forces r as

$$r_{i,j}^{(p)}=h^2+\frac{1}{4}\left\{w_{i+1,j}^{(p)}+w_{i-1,j}^{(p+1)} -4w_{i,j}^{(p)}+w_{i,j+1}^{(p)}+w_{i,j-1}^{(p+1)}\right\} \quad (10)$$

Then, Eq. (9) can be re-written using the residual forces r as

$$w_{i,j}^{(p+1)}=w_{i,j}^{(p)}+r_{i,j}^{(p)} \quad (11)$$

When the residual forces become negligible, the solution satisfy the governing equation. Various acceleration technique can be adopted to accelerate the iteration process. One of the most efficient method is to use an optimum over relaxation factor, α_{opt}, as

$$w_{i,j}^{(p+1)}=w_{i,j}^{(p)}+\alpha_{opt}r_{i,j}^{(p)} \quad (12)$$

where $\alpha_{opt}=2/\{1+\sqrt{1-\lambda}\}$,

and $\lambda^2=\sum_{i=1}^{n}\{r_i^{(p)}\}^2/\sum_{i=1}^{n}r_i^{(p)}\,r_i^{(p+1)}$

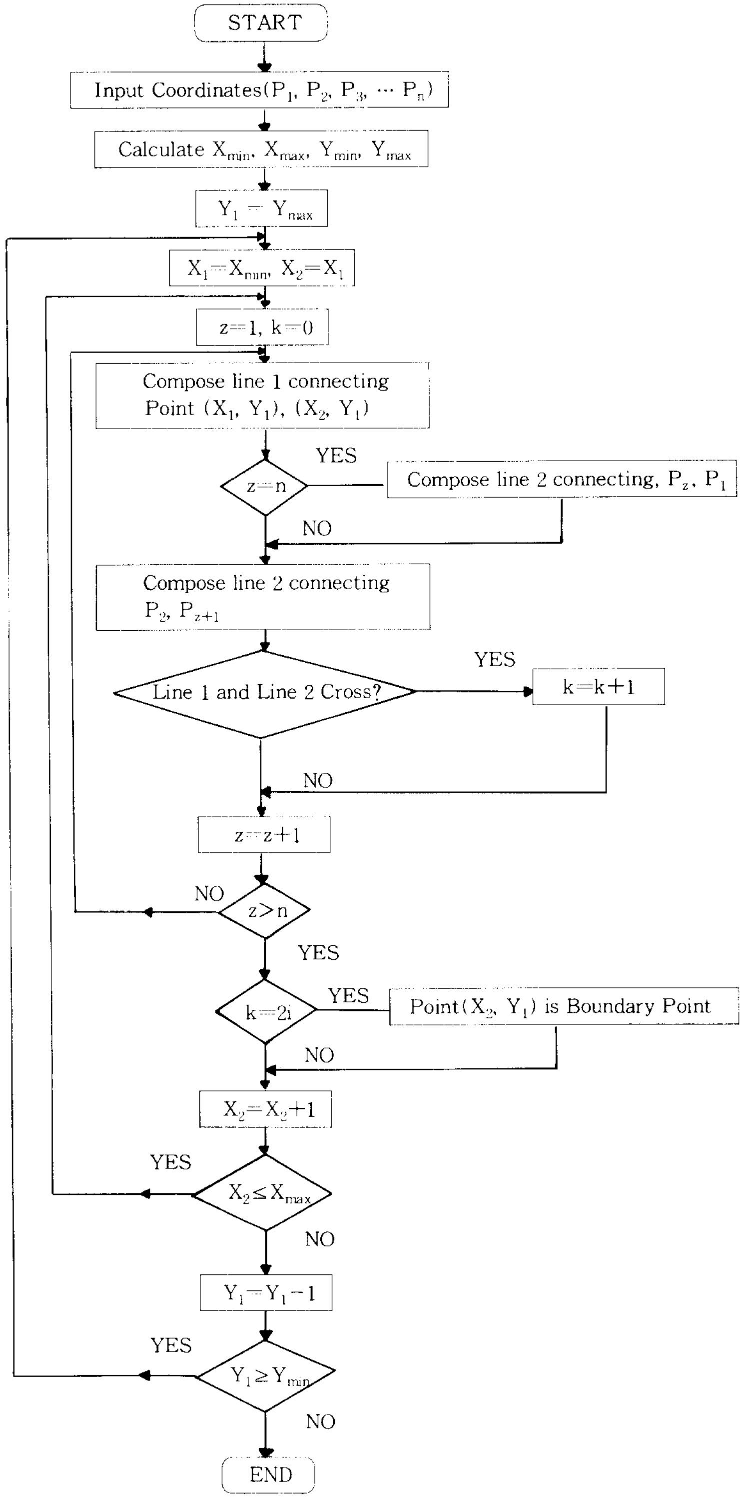

Fig. 1 Flow chart of automatic boundary generating method

We call this process as *Successive Over Relaxation Method (SOR)* which combines successive displacement with over relaxation method. This method accelerates the convergence rate significantly. In case of section with holes, the boundary condition of Eq. (4) was discretized after assuming q = 4 S as before.

$$\left(\sum_{i=1}^{n} w_i - n\, w_0\right) + 4A = 0 \qquad (13)$$

where n = the number of strings attaching the area of the hole to the rest of the net, w_i = the deflection of a nodal point i adjacent to the hole, w_0 = the deflection of the boundary of the hole and A = the area of the hole.

4. Automatic Boundary Generating Method

Computer program TORSION has been developed to satisfy Eq. (12) with the boundary conditions, i. e. , zero for the solid section and Eq. (13) for the hollow section. An identifying the boundary values of the rectangular and circular section is relatively simple but those of the general shaped section require cumbersome efforts. For the simple input process of the boundary condition, the algorithm of ***Automatic Boundary Generating Method*** has been developed. The Automatic Boundary Generating Method (ABGM) starts with using the characteristics of the intersection of the lines. The maximum and minimum x,y coordinates are decided using the vertices coordinates. Then, grid lines x,y direction each are composed using the points of minimum and maximum. Generated grid lines in one direction will intersect with the boundary line of the section and the total number of intersection becomes 2i, where i = 1, 2, 3··· depending on the shape. The intersection coordinates will be stored and was checked using the other directional grid lines. A detail flow chart of the ABGM was shown in Fig. 1. Using this technique the boundary coordinates in any section could be found without any difficulty.

5. Numerical Example — Rectangular Solid Section

As a simple example, torsion constant, K_T, and maximum shear stress, τ_{max}, were calculated for the same problem introduced by Hsu[1] who used St. Venant's semi-inversive method. He calculated St. Venant's coefficients for rectangular sections with different side ratios (=d/b, d=the large and b=the small dimension of rectangular cross section) ranging from 1.0 to 100 using eight terms and eight significant figures. He also showed the coefficients of the maximum shear stress to the torque. Comparison of the results from the present study with those from Hsu[1] were made in Table 1 and the agreements were excellent. A typical solid rectangular cross section with d=100 cm and b=50 cm was studied to show

Table 1 Comparison of the results for α and β

d/b	β		α	
	St. Venant(1)	Present Study	St. Venant(1)	Present Study
1.0	0.141	0.14053	0.208	0.20970
1.2	0.166	0.16606	0.219	0.22035
1.4	0.187	0.18683	0.227	0.22870
1.6	0.204	0.20365	0.234	0.23565
1.8	0.217	0.21734	0.240	0.24173
2.0	0.229	0.22864	0.246	0.24718
2.5	0.249	0.24927	0.258	0.25891
3.0	0.264	0.26328	0.267	0.26855
4.0	0.281	0.28077	0.282	0.28307
5.0	0.291	0.29128	0.291	0.29295
10.0	0.312	0.31229	0.312	0.31387
100.0	0.331	0.33085	0.331	0.33453

Note : $K_T = \beta b^3 d$, $\tau_{max} = \dfrac{T}{\alpha b^2 d}$

Table 2 Comparison of the iteration numbers

Method	Iteration Number	Convergence Criteria
Simultaneous	9404	1.0E 09
Successive	4730	1.0E 09
SOR	679	1.0E 09

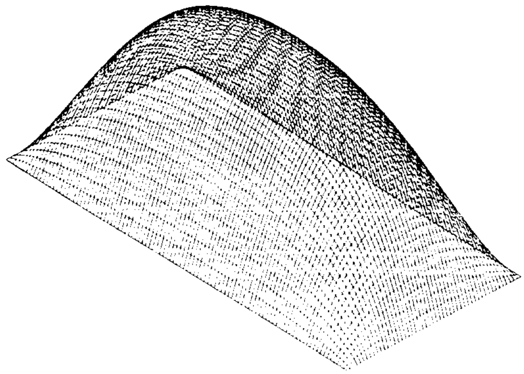

Fig. 2 Deflected Membrane Shape

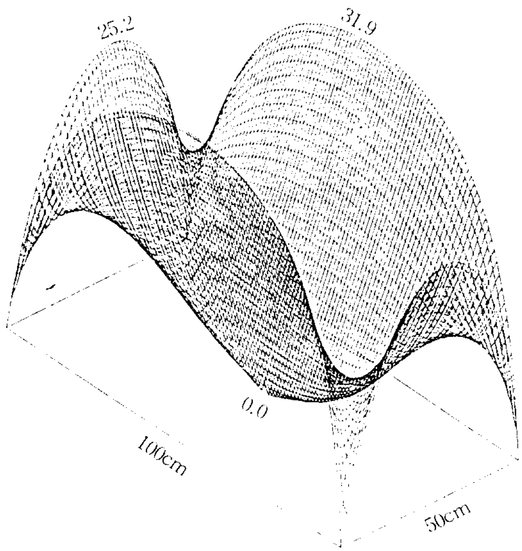

Fig. 3 Stress contour(kg/cm^2)

the effectiveness of the Successive Over Relaxation Method. Three different iteration methods such as Simultaneous Iteration, Successive Iteration and successive Over Relaxation (SOR) were used and the number of iterations from each methods were compared as shown in Table 2. The deflected membrane shape and the stress contour using torsional moment of 20 ton-m were shown in Fig. 2 and Fig. 3, respectively.

6. Numerical Example—Hollow Square Section

The torsion problem for a hollow square section was solved by Southwell[5]. He solved this problem for one eighth of the section because of the geometric symmetry as shown in Fig. 4a. He assumed the outer and inner boundary values as zero and 1210, respectively and solved the torsional membrane equation using Relaxation Method. We used the same boundary values in this study and solved Prandtl's membrane equation using the optimum Over Relaxation factor. The result from this study and those by Southwell[5] showed a very close agreement as shown in Figs. 4. The deflected shape and the stress contour of the hollow square section showed in Figs. 5 and 6 using torsional moment of 20 ton-m. Fig. 6 also shows that the stress concentration factor at the re-entrant corners is approximately 1.7 which is comparable to those calculated by Southwell[5]. In this case, the radius of the fillet was 10 % of the thickness.

Torsion problem of hollow section with double cells was also solved using computer program TORSION. The deflected shape of the membrane and the stress contour were shown in Figs. 7 and 8, respectively using torsional moment of 20 ton-m. It shows that the stress concentration at the corner without a fillet is very high.

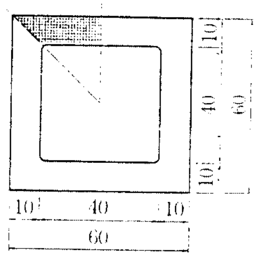

Fig. 4(a) Hollow square section(cm)

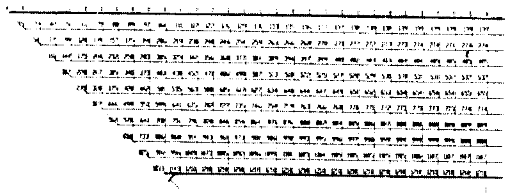

Fig. 4(b) The height of membrane by southwell[5]

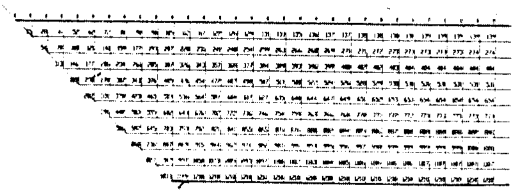

Fig. 4(c) The height of membrane from this study

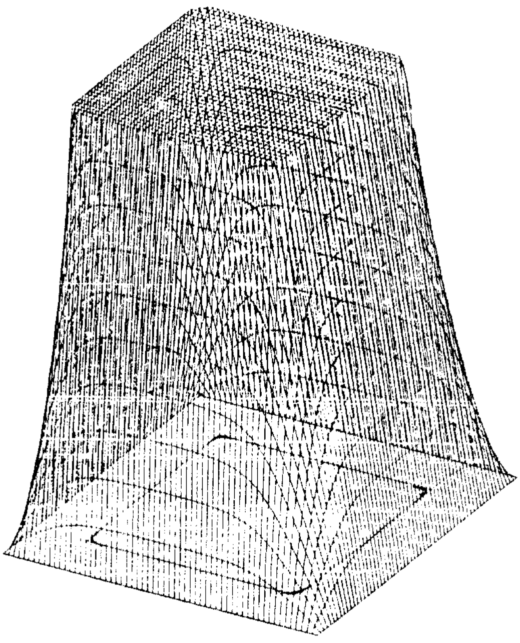

Fig. 5 Deflected membrane shape

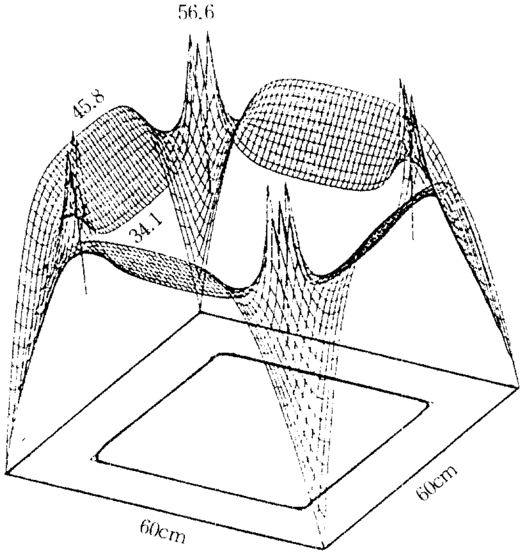

Fig. 6 Stress contour(kg/cm²)

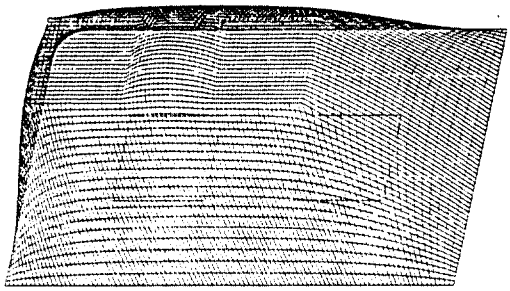

Fig. 7 Deflected membrane shape

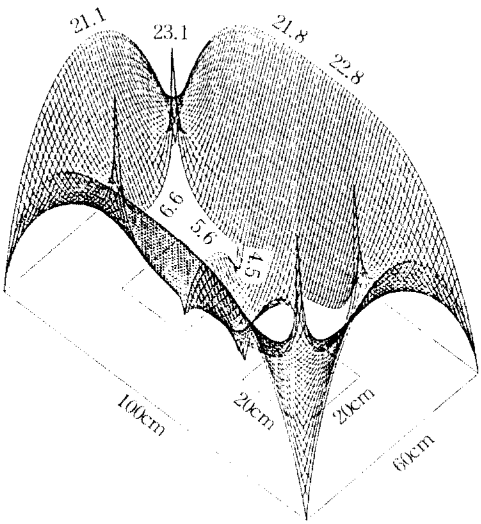

Fig. 8 Stress contour(kg/cm²)

7. Approximate formulas of Torsion Constant for Hollow Section

Practically in most of concrete structure a hollow section such as circular, rectangular, elliptical, etc. is inevitable to reduce the hydration heat during the curing process. The evaluation of the torsion constant of the hollow section, however, is ambiguous due to the two different approximate methods. The torsion constant of a thick-hollow section can be found by calculating those for the shape of the outside boundary and deducting the value calculated for the inside boundary using the data from Table 1. The torsion constant of a thin-walled hollow section, however, can be evaluated using the approximate formula proposed by Bredt[1, 2] as

$$K_T = \frac{4A^2}{\oint ds/t} \tag{14}$$

where A is the area enclosed by the center line of the walls, shaded area in Fig. 9 and $\oint ds/t$ is the integral round the wall center line of the length divided by the wall thickness. The question comes right away how thin the wall to apply Bredt's equation and the judgement only depends on the past experiences or intuitions.

When any thin-walled hollow section is transformed into an equivalent circular hollow section which has same torsion area, the torsion constant will not be changed since the torsion constant of a thin-walled hollow section is a function of the same variable A. The equivalent radius a and the equivalent thickness t_{eq} can be formulated as

$$a = \sqrt{\frac{A}{\pi}} \text{ and } t_{eq} = \frac{\oint t\,ds}{2\sqrt{\pi A}} \tag{15}$$

where a is the radius of the equivalent circular section from the center to the center line of the wall, and t_{eq} is the equivalent thickness of the wall as shown in Fig. 9.

Since Bredt's equation assumed an uniform shear stress through the section, the error becomes large as the wall thickness increases. Using the computer program TORSION, the torsion constant for various hollow sections with different wall thickness were calculated and compared with those from the approximate thin-walled formula shown in Eq. (14) for the various non-dimensional constant a/t_{eq}. The magnitude of error from various cases for a/t_{eq} was bounded to the error function. An equilateral hyperbola type error function, then, was obtained up to the certain range as shown in Fig. 10.

$$\text{Error}(\%) = 100 - 30(t_{eq}/a)^{3/2} \quad : a/t_{eq} \geq 1.5 \tag{16}$$

Now, the approximate formulation of the torsion constant for any hollow section are obtained combining Eqs. (14) and (16) for the range of $a/t_{eq} \geq 1.5$.

$$K_T = \text{Thick-Hollow Section Theory} \quad : a/t_{eq} \leq 1.5$$
$$= \frac{4A^2}{\oint ds/t}\{1 - 0.30(t_{eq}/a)^{3/2}\}^{-1} \quad : a/t_{eq} \geq 1.5 \tag{17}$$

where Thick-Hollow Section Theory evaluates the torsion constant by assuming the solid section using the outside boundary and deducting the value using the inside boundary.

The torsion constants for the three different rectangular hollow sections were calculated using the approximate formulas and compared with those from the exact values using the program TORSION and from the Korean Concrete Standard[7]. Figs. 11 showed that the proposed approximate formulas gave much better solution than the Korean Concrete Standard.

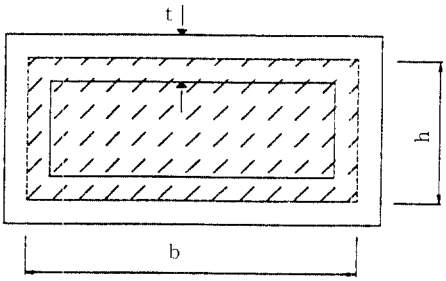

Fig. 9(a) Hollow rectangular section

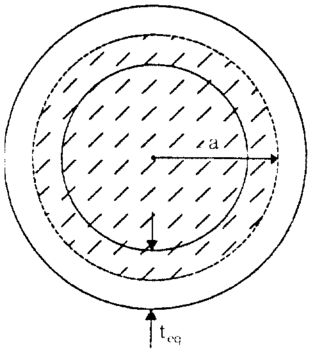

Fig. 9(b) Equivalent circular section

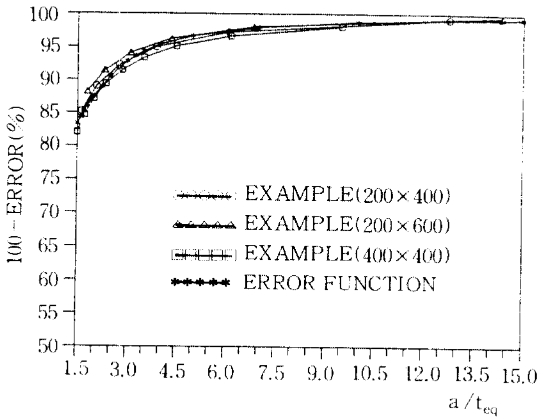

Fig. 10 Error function for the various sections

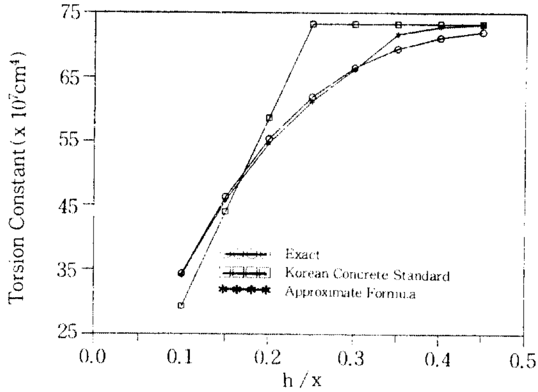

Fig. 11(a) Comparison of torsion constant(200×400cm) where h=thickness, x=shorter side length

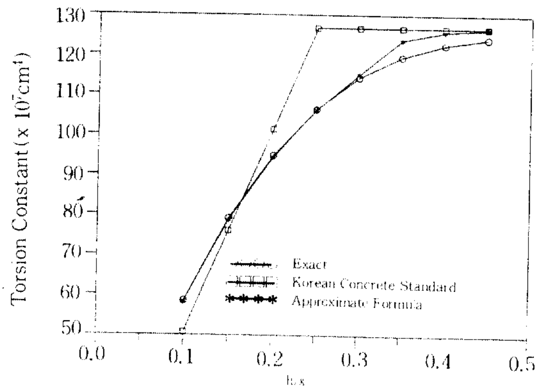

Fig. 11(b) Comparison of torsion constant(200×600cm) where h=thickness, x=shorter side length

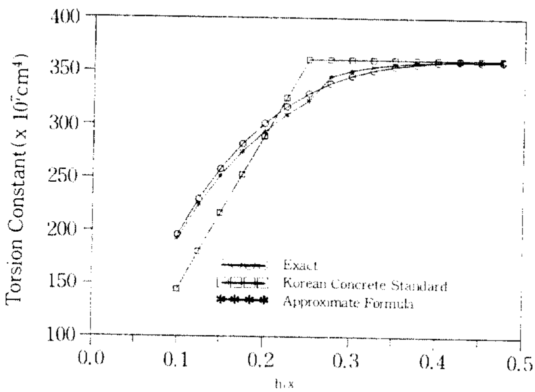

Fig. 11(c) Comparison of torsion constant(400×400cm) where h=thickness, x=shorter side length

8. Conclusions

General computer program TORSION was developed and proved to be accurate to calculate a torsion constant and shear stress for any section shapes with or without holes. Successive Over Relaxation Method was very efficient to accelerate the iteration process for the finite difference method. To relieve the cumbersome efforts to prepare the input values of boundary for the general shaped section, Automatic Boundary Generating Method was developed and applied effectively. The approximate formulas to calculate the torsion constant for the general shaped hollow section were very simple and gave much better solution than the Korean Concrete Standard.

9. References

1. Hsu, T. C., Torsion of Reinforced Concrete, Van Nostrand Reinhold Co., 1984, pp.3-49
2. Timoshenko, S. P. and Goodier, J. N., Theory of Elasticity, 2nd ed., McGraw-Hill Book Co., N. Y., 1970, pp.291-349
3. Cook, R. D. and Young, W. C., Advanced Mechanics of Materials, Macmillian Publishing Co., N. Y., 1985, pp.283-333
4. Heins, C. P., Bending and Torsional Design in STructural Members, Lexington Books, 1975, pp.36-64.
5. Southwell, R. V., Relaxation Methods in Theoretical Physics, Oxford University Press, Fair Lawn, N. J., 1946, pp.60
6. Bickford, W. B., A first Course in the Finite Element Method, IRWIN, INC., 1990, pp.333
7. Korean Concrete Standard, Korean Ministry of Construction, 1988

(접수일자 : 1992. 11. 22)