복합재료적층판의 진동해석을 위한 유한요소모델
II. 유한요소모델의 유도 및 해석

Finite Element Analysis for Vibration of Laminated Plate Using a Consistent Discrete Theory
Part II : Finite Element Formulation and Implementations

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요 약

앞의 논문 Part I에서 유도한 변분원리를 이용하여 복합재료적층판의 진동해석을 할 수 있는 유한요소해석 모델을 개발하였다. 이 모델에서는 어느 한 층의 면내 변위와 나머지층 단면의 회전각, 그리고 판 전체의 연직방향처짐을 절점변수로 취하게 되어 n개층으로된 적층판의 경우 2(n+1)+1의 절점 자유도를 갖는다. 따라서, 판의 주변에서는 한층의 면내변위와 각층단면의 회전각을, 판의 면내에서는 연직방향 처짐을 경계조건값으로 정의할 수 있다. 이 모델에 의해 개발한 프로그램을 이용하여 각층의 재료특성이 크게 다른 혼종형 복합재료적층판(hybrid laminate)의 고유진동문제를 해석하였다. 탄성이론해 및 다른 유한요소해석결과와 본 해석결과와의 비교를 통해 제시모델이 기존의 다른 유한요소모델보다 정확함을 예시하였다.

Abstract

Based on a variational principle of the consistent shear deformable discrete laminate theory derived in the companion paper Part I, a finite element procedure for the vibration analysis of laminated composite plates is presented. The present formulation takes the in-plane displacements of an arbitrary layer, the rotations of the cross section of each layer and tranverse displacement of the plate as the state variables at a nodal point of finite element, resulting in total nodal degree of freedom of 2(n+1)+1 for the n-layered laminate. Thus, it allows to specify displacement boundary conditions of layer stretching and/or rotation of layer cross sections around the plate edge and/or lateral displacement. The developed procedure is applied to the free vibration problem for sandwich-type hybrid laminates composed of layers with drastically different material properties whose elasticity solutions are known. Comparison of analysis results with other FEM solutions showed that the present formulation yields better accuracy.

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1. INTRODUCTION

In the design of structures composed of advanced fiber-reinforced composite laminates, it is essential to predict mechanical behavior accurately. Analytical solutions of the elasticity equations or plate equations have been obtained for some static and dynamic problems with simple lamination scheme and geometry[1-6]. For laminated plates with arbitrary stacking sequence and irregular configuration, however, the problem becomes intractable due to the complexity of its governing equations. This has motivated a considerable research effort to develop efficient and reliable numerical solution techniques such as finite element procedure.

Since Pryor and Barker[7] initiated the development of finite element procedure for the analysis of laminated plates, an enormous effort has been made to propose various FEM procedures using different techniques as well as laminate theories. Mawenya and Davis[8] and Panda and Natarajan[9] used the quadratic shell elements of Ahmad et al.[10] to analyze the bending of multilayer plates. Mau et al.[11, 12] and Spilker et al.[13] applied the so-called hybrid formulations to the static and dynamic analysis of laminated plates. Mixed formulations were also presented by Noor and Mathers[14, 15] for the study of vibration problems. Reddy[16] applied penalty method to the static and dynamic analysis of laminated composite plates. However, all these works were based on the first order shear deformable theory(FSDT) by Yang et al.[17], which has an intrinsic deficiency in incorporating local shear deformation accurately, resulting in significant error when shear rigidities of adjacent layers are much different.

To remove the error from deficiencies of FSDT in stress resolution as well as global behaviour, more refined theories have been utilized as the basis for finite element formulations. Using a simple higher-order laminate theory[18], Putcha and Reddy[19] presented a mixed-type FEM procedure for the dynamic and stability analysis of laminated composites. Reddy et al.[20] also developed a finite element procedure based on Srinivas' discrete laminate theory[21] and studied dynamic characteristics of laminated plates. However, Srinivas' theory does not ensure the interlaminar continuity of the transverse shear stresses. Recently, Peseux and Dubugeon[22] and Liau and Tsai[23] applied Hellinger-Reissner mixed variational functional to develop the finite element procedures which satisfy the transverse shear stress continuity at the layer interface. These procedures with 5N nodal degree of freedom where N is number of layers appeared to yield reasonably good shear stress resolutions. For the thick laminated with a large number of layers, Zywicz[24] proposed a homogenization method of material properties, in which laminate stiffness is represented with polynomial functions so as to account for the layer moduli, but continuities of the transverse shear stresses and in-plane displacements are satisfied. Advantage of this approach is that finite element has constant number of nodal degree of freedom regardless number of layers. In addition, Lee and Liu[25] used Pagano's so-called 'Global-Local model'[26] for the finite element formulation, but the results appeared to be 'not really exact' in predicting transverse shear stress distribution through thickness of a laminate.

In this paper, we present a finite element procedure for the dynamic analysis of laminated plates based on the discrete laminated plate theory which incorporates the effect of transverse shear deformations in a variationally consistent manner[27]. As the basis of finite element formulation, a variational principle derived in the companion paper[28] is used. The developed procedure is applied for the analysis of vibration characteristics of laminated composite plates whose elasticity solutions are available and the results are compared with other works.

2. VARIATIONAL PRINCIPLE

A usual finite element procedure for the solution of initial-boundary value problem is to discretize the spatial domain by finite element method and to use other technique, e.g., direct integration or mode superposition method, to solve resulting discretized system of equations in time domain. In connection with spatial discretization, the governing functional needs to be defined on each element and summed up for all elements for the finite element representation.

Over an element, the govening functional Ω_3 given in Ref.[28], which is defined in terms of displacement field variables can be written as

$$
\begin{aligned}
\Omega_e = & -\sum_{k=1}^{n}\{ <\bar{u}^1_\alpha, P^k\bar{u}^1_\alpha>_{R^k_e} + 2<\bar{u}^1_\alpha, P^k\sum_{i=1}^{k-1}t_i\phi^i>_{R^k_e} + <\sum_{i=1}^{k-1}t_i\phi^i_\alpha, P^k\sum_{j=1}^{k-1}t_j\phi^j_\alpha>_{R^k_e} \\
& +<\phi^k_\alpha, I^k\phi^k_\alpha>_{R^k_e} + <w, P^k w>_{R^k_e} + 2<\bar{u}^1_\alpha, R^k\phi^k_\alpha>_{R^k_e} \\
& +2<\sum_{j=1}^{k-1}t_j\phi^i_\alpha, R^k\phi^k_\alpha>_{R^k_e} + <\bar{\varepsilon}^1_{\alpha\beta}, t^* A^k_{\alpha\beta\gamma\delta}\bar{\varepsilon}^1_{\gamma\delta}>_{R^k_e} \\
& +2<\bar{\varepsilon}^1_{\alpha\beta}, t^* A^k_{\alpha\beta\gamma\delta}\sum_{i=1}^{k-1}t_i\kappa^i_{\gamma\delta}>_{R^k_e} + <\sum_{i=1}^{k-1}t_i\kappa^k_{\alpha\beta}, t^* A^k_{\alpha\beta\gamma\delta}\sum_{j=1}^{k-1}t_j\kappa^j_{\gamma\delta}>_{R^k_e} \\
& +<\kappa^k_{\alpha\beta}, t^* D^k_{\alpha\beta\gamma\delta}\kappa^k_{\gamma\varepsilon}>_{R^k_e} + 2<\bar{\varepsilon}^1_{\alpha\beta}, t^* B^k_{\alpha\beta\gamma\delta}\kappa^k_{\gamma\delta}>_{R^k_e} \\
& +2<\sum_{i=1}^{k-1}t_i\kappa^i_{\alpha\beta}, t^* B^k_{\alpha\beta\gamma\delta}\kappa^k_{\gamma\delta}>R^k_e\} \\
& -\sum_{k=1}^{n}\sum_{j=1}^{n} <2e^k_{\alpha 3}, 2t^*\lambda^{kj}_{\alpha\beta}e^j_{\beta 3}>_{R^k_e} \\
& +2\sum_{k=1}^{n}\{ <\bar{u}^1_\alpha, t^* F^k_\alpha + X^k_\alpha>_{R^k_e} + <\sum_{i=1}^{k-1}t_i\phi^i_\alpha, t^* F^k_\alpha + X^k_\alpha>_{R^k_e} \\
& +<\phi^k_\alpha, t^* G^k_\alpha + Y^k_\alpha>_{R^k_e} + <w, t^* F^k_3 + Z^k>_{R^k_e}\} \\
& -2<\bar{u}^1_\alpha, t^* T^0_\alpha>_{R^1_e} - 2<w, t^* T^0_3>_{R^1_e} + 2<\bar{u}^1_\alpha, t^* T^n_\alpha>_{R^n_e} \\
& +2<\sum_{i=1}^{n}t_i\phi^i_\alpha, t^* T^n_\alpha>_{R^n_e} + 2<w, t^* T^n_3>_{R^n_e} \\
& -2\sum_{k=1}^{n}\{ <\bar{u}^1_\alpha, t^*\hat{N}^k_\alpha>_{S^k_{1e}} + <\sum_{i=1}^{k-1}t_i\phi^i_\alpha, t^*\hat{N}^k_\alpha>_{S^k_{1e}} + <\phi^k_\alpha, t^*\hat{M}^k_\alpha>_{S^k_{5e}} \\
& +<w, t^*\hat{Q}^k_\alpha\eta_\alpha>_{S^k_{5e}}\}
\end{aligned}
\tag{1}
$$

where S^k_{1e}, S^k_{2e}, and S^k_{5e} are the intersections of the boundary surface of an element with S^k_1, S^k_2 and S^k_5, respectively. It is reminded that this functional satisfies the kinematic relations and displacement continuities in laminar surfaces.

For spatial discretization of this functional, it is assumed that the field variables are interpolated over an element domain as

$$
\begin{aligned}
\bar{u}^1(x,t) &= \mathbf{H}^T_u(x)\mathbf{U}(t) \\
\phi^k(x,t) &= \mathbf{H}^T_\phi(x)\Phi^k(t) \\
w(x,t) &= \mathbf{H}^T_w(x)\mathbf{W}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{u}^1 &= \left[\bar{u}^1_1,\ \bar{u}^1_2\right]^T \\
\phi^k &= \left[\phi^k_1,\ \phi^k_2\right]^T
\end{aligned}
\tag{2}
$$

and U(t), $\Phi^k(t)$, W(t) are the vector functions of time defined at the nodal points and H_u, H_ϕ, H_w, respectively, the matrices of spatial interpolation functions for the field variables indicated by subscripts. Also, the generalized strains may be expressed as

$$
\begin{aligned}
& T^T_u U(t) \\
&= T^T_\phi \Phi^k(t) \\
&= T^I_s W(t) + \mathbf{H}^T_\phi \Phi^k(t)
\end{aligned}
$$

where

$$\left[\bar{e}^1_{11},\ \bar{e}^1_{22},\ \bar{e}^1_{12}\right]^T$$
$$\left[\kappa^k_{11},\ \kappa^k_{22},\ 2\kappa^k_{12}\right]^T$$
$$2\left[e^k_{23},\ e^k_{13}\right]^T \tag{3}$$

and T_u, T_ϕ and T_s are, respectively, the transformation matrices derived from the displacement cement interpolation functions H_u, H_ϕ and H_w by suitable differentiation and reorganization of terms.

Substituting (2) and (3) into (1), the spatially discretized governing function is obtained.

$$\begin{aligned}
\Omega_e = &-\sum_{k=1}^{r}\Big\{U^T M^k_{uu} * U + U^T M^k_{u\phi\phi} * \sum_{i=1}^{k-1} t_i\,\Phi^i + \sum_{i=1}^{k-1} t_i(\Phi^i)^T M^k_{\phi\phi p} * \sum_{j=1}^{k-1} t_i\Phi^j \\
&+(\Phi^k)^T M^k_{\phi\phi I} * \Phi^k + W^T M^k_{ww} * W + 2U^T M^k_{U\phi R} * \Phi^k \\
&+2\sum_{i=1}^{k-1} t_i(\Phi^i)^k M^k_{\phi\phi R} * \Phi^k + t * U^T K^k_{uuA} * U + 2t * U^T K^k_{u\phi A} * \sum_{i=1}^{k-1} t_i\Phi^i \\
&+t * \sum_{i=1}^{k-1} t_i(\Phi^i)^T K^k_{\phi\phi A} * \sum_{j=1}^{k-1} t_j\Phi^j + t * (\Phi^k)^T K^k_{\phi\phi D} * \Phi^k \\
&+2t * U^T K^k_{u\phi B} * \Phi^k + 2t * \sum_{i=1}^{k-1} t_i(\Phi^i)^T K^k_{\phi\phi B} * \Phi^k\Big\} \\
&-t * \sum_{k=1}^{n}\Big\{W^T \sum_{j=1}^{n} K^{kj}_{sww} * \Phi^j + 2W^T \sum_{j=1}^{n} K^{kj}_{sw\phi} * \Phi^j + (\Phi^k)^T \sum_{j=1}^{n} K^{kj}_{s\phi\phi} * \Phi^j\Big\} \\
&+2\sum_{k=1}^{r}\Big\{t * U^T * f^k + U^T * b^k_s + t * \sum_{i=1}^{k-1} t_i(\Phi^i)^T * h^k + (\Phi^i)^T * r^k_s \\
&+t * (\Phi^k)^T * g^k + (\Phi^k)^T * b^k_s + t * W^T * f^k_3 + W^T * b^k_z\Big\} \\
&+2t * \Big\{-U^T * \tau^c + W^T * (P^n - P^o) + U^T * \tau^n + \sum_{i=1}^{n} t_i(\Phi^i)^T * \tau^n\Big\} \\
&-2t * \sum_{k=1}^{n}\Big\{U^T * R^k_{n1} + \sum_{i=1}^{k-1} t_i(\Phi^i)^T * R^k_{n2} + (\Phi^k)^T * R^k_m + W^T * R^k_q\Big\}
\end{aligned} \tag{4}$$

Definitions of matrices in this spatially discretized functional are given in Appendix A.

3. SEMIDISCRETE EQUATIONS OF MOTION

To obtain the semidiscrete equations of motion of laminated plate, it is convenient to write the spatially discretized variational functional (4) in matrix form as

$$\Omega_e = -X_e^T S_e * X_e + 2X_e^T * R_e \tag{5}$$

where

$$X_e = \begin{Bmatrix} U \\ - \\ \Phi^1 \\ \Phi^2 \\ \cdot \\ \cdot \\ \cdot \\ \Phi^n \\ - \\ W \end{Bmatrix} \qquad R_e = \begin{Bmatrix} P_u \\ - \\ P_{\phi^1} \\ P_{\phi^2} \\ \cdot \\ \cdot \\ \cdot \\ P_{\phi^n} \\ - \\ P_w \end{Bmatrix} \qquad S_e = \begin{bmatrix} S_{11} & | & S_{12} & | & 0 \\ - & | & - & | & - \\ S^T_{12} & | & S_{22} & | & S_{23} \\ - & | & - & | & - \\ 0 & | & S^T_{23} & | & S_{33} \end{bmatrix}$$

Here, element of the submatrices of S_e and the load vector R_e are explicitly given in Appendix B. From Eq.(5), it is seen that the nodal degree of freedom for a laminate with n layers is equal to 2(n+1)+1. The spatially discretized variational functional of global system is given by

$$\Omega = \sum_{e=1}^{m} \Omega_e = -X^T S * X + 2X^T * R \tag{6}$$

where X is the vector of values of field variables at the system nodal points, R is the set of corresponding "forcing" quantities and S is the system matrix corresponding to S_e for an element. Here, summation in (6) is not algebraic sum, but rather indicates the matrix assembly following usual procedure. Vanishing of the differential of Ω in (6) with respect to X gives the set of equations.

$$SX = MX + t^* KX = R \tag{7}$$

where

$$M = \begin{vmatrix} \sum_{k=1}^{n} M^k_{uu} & t_1\sum_{k=2}^{n} M^k_{u\phi p} + M^1_{u\phi R} & t_2\sum_{k=3}^{n} M^k_{u\phi p} + M^2_{u\phi R} & t_3\sum_{k=4}^{n} M^k_{u\phi p} + M^3_{u\phi R} & \cdots \\ & t_1^2\sum_{k=2}^{n} M^k_{\phi\phi p} + M^1_{\phi\phi I} & t_1 t_2\sum_{k=3}^{n} M^k_{\phi\phi p} + t_1 M^2_{\phi\phi R} & t_1 t_3\sum_{k=4}^{n} M^k_{\phi\phi p} + t_1 M^3_{\phi\phi R} & \cdots \\ & & t_2^2\sum_{k=3}^{n} M^k_{\phi\phi p} + M^2_{\phi\phi I} & t_2 t_3\sum_{k=4}^{n} M^k_{\phi\phi p} + t_2 M^3_{\phi\phi R} & \cdots \\ & & & t_3^2\sum_{k=4}^{n} M^k_{\phi\phi p} + M^3_{\phi\phi I} & \cdots \end{vmatrix}$$

$$
\begin{array}{lllll}
\cdots\; t_{n-2}\sum_{k=n-1}^{n} M_{u\phi p}^{k} + M_{u\phi R}^{n-2} & t_{n-1}\sum_{k=n}^{n} M_{u\phi p}^{k} + M_{u\phi R}^{n-1} & M_{u\phi R}^{n} & 0 \\
\cdots\; t_1 t_{n-2}\sum_{k=n-1}^{n} M_{\phi\phi p}^{k} + t_1 M_{\phi\phi R}^{n-2} & t_1 t_{n-1}\sum_{k=n}^{n} M_{\phi\phi p}^{k} + t_1 M_{\phi\phi R}^{n-1} & t_1 M_{\phi\phi R}^{n} & 0 \\
\cdots\; t_2 t_{n-2}\sum_{k=n-1}^{n} M_{\phi\phi p}^{k} + t_2 M_{\phi\phi R}^{n-2} & t_2 t_{n-1}\sum_{k=n}^{n} M_{\phi\phi p}^{k} + t_2 M_{\phi\phi R}^{n-1} & t_2 M_{\phi\phi R}^{n} & 0 \\
\cdots\; t_3 t_{n-2}\sum_{k=n-1}^{n} M_{\phi\phi p}^{k} + t_3 M_{\phi\phi R}^{n-2} & t_3 t_{n-1}\sum_{k=n}^{n} M_{\phi\phi p}^{k} + t_2 M_{\phi\phi R}^{n-2} & t_3 M_{\phi\phi R}^{n} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
t_{n-2}^{2}\sum_{k=n-1}^{n} M_{\phi\phi p}^{k} + M_{\phi\phi I}^{n-2} & t_{n-2} t_{n-1}\sum_{k=n}^{n} M_{\phi\phi p}^{k} + t_{n-2} M_{\phi\phi R}^{n-1} & t_{n-2} M_{\phi\phi R}^{n} & 0 \\
 & t_{n-1}^{2}\sum_{k=n}^{n} M_{\phi\phi p}^{k} + M_{\phi\phi I}^{n-1} & t_{n-1} M_{\phi\phi R}^{n} & 0 \\
 & & M_{\phi\phi I}^{n} & 0 \\
 & & & \sum_{k=1}^{n} M_{ww}^{k}
\end{array}
$$

Differentiating (7) twice with respect to time, we obtain the semidiscrete equations of motion.

$$M\ddot{X} + KX = R \qquad (8)$$

where M, K are, respectively, mass and stiffness matrices, and superposed dot denotes time derivative. In this equation, inherent material damping was not allowed for and may be introduced in terms of equivalent viscous damping which is proportional to mass and stiffness matrices. Also, it is worthwhile to note that Eq.(8) reduces to the equations for the static case if the inertial term is omitted.

4. NUMERICAL IMPLEMENTATIONS

A finite element code was developed using the proposed formulation and applied for the analysis of vibration characteristics of laminated composite plates to test its performance. The code incorporated the "Heterosis" element due to Hughes and Cohen[29], which is a kind of composite element employing eight-node serendipity and nine-node Lagrange interpolation functions for the field variables w and u_x, ϕ_x, respectively. To avoid "shear locking" phenomenon, selective/reduced Gaussian integration was utilized.

The first problem tested was a simply supported sandwich-type orthotropic laminated plate composed of three layers as shown in Fig. 1, whose elasticity solution was given by Srinivas and Rao[4]. Since the plate had biaxial symmetry with respect to x_1− and x_2-axis, only one quadrant of the platform was discretized and shown along with boundary conditions in Fig. 2. Top and bottom layers were assumed to have the same thickness and material properties while the thickness and material properties of the middle layer are different. For this laminate, three different sets of material properties given in Table 1 were considered, in which the ratio of reduced elastic constants[30] of the middle layer to those of the outer layers was 1, 10 and 50, respectively.

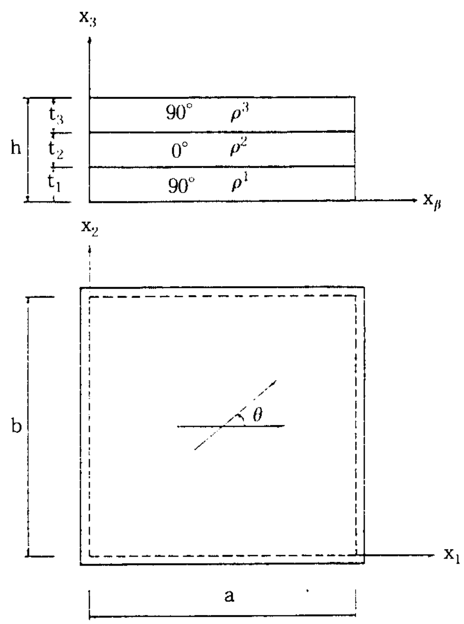

Fig. 1 Platform and Cross Section of laminated Plate

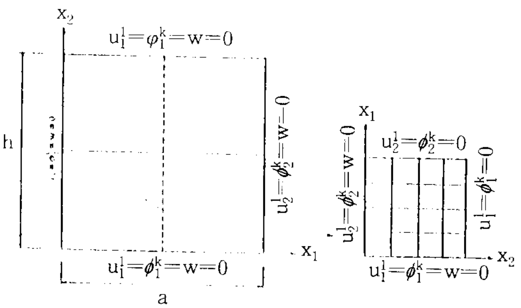

Fig. 2 Finite Element Model of a Quadrant of Plate

Table 1. Lamination Data for Numerical Test

CASE	t_1/h	t_2/h	t_3/h	ρ^1/ρ^2	ρ^3/ρ^2	Q^1_{66}/Q^2_{66}	Q^3_{66}/Q^2_{66}
I	0.1	0.8	0.1	0.1	1.0	1	1
II	0.1	0.8	0.1	0.1	1.0	10	10
III	0.1	0.8	0.1	0.1	1.0	50	50

* For all layers, ratios of orthotropic elastic constants were :

$\bar{Q}_{11} : \bar{Q}_{12} : \bar{Q}_{22} : \bar{Q}_{44} : \bar{Q}_{55} : \bar{Q}_{66}$
$=3.802:0.879:1.996:1.015:0.608:1.0$

Table 2 shows the non-dimensionalized fundamental frequency obtained using 4-, 16- and 36-element mesh. For comparison, elasticity solutions and the finite element solutions based on Srinivas theory[21] with the shear correction factor k=5/6 were given together. With 16-element mesh, the present results are in good agreement with elasticity solutions, showing percentage error was 0.5%, 1.8% and 3.2% for the case I, II and III, respectively. The 16-element mesh yielded much improved results over the 4 element mesh, but there was no noticeable improvement by 36 element mesh over 16 element mesh.

From the comparison of finite element solutions based on the present formulation and Srinivas' theory, it is seen that the latter overpredictes natural frequency more than the former. Difference increases as the difference in stiffness of the outer layers and the inner layer grows. This implies that effect of the consistent incorporation of shear deformation in numerical formulation becomes more pronounced where the difference in material properties between layers is large such as hybrid laminates.

Table 2. Non-dimensionalized fundamental frequency λ by : (a) present formulation, (b) Srinivas' laminate theory, (c) elasticity theory

Mesh		CASE I		CASE II		CASE III	
			Error		Error		Error
4	(a)	0.094697	2.4%	0.196820	2.9%	0.310970	3.8%
	(b)	0.094980	2.7%	0.197100	3.0%	0.314980	5.1%
16	(a)	0.092592	0.5%	0.194720	1.8%	0.309190	3.2%
	(b)	0.093170	0.7%	0.195840	2.3%	0.313050	4.5%
36	(a)	0.092900	0.4%	0.194630	1.7%	0.309110	3.1%
	(b)	0.093410	0.7%	0.198050	3.5%	0.322320	7.6%
(c)		0.092480		0.191320		0.299540	

* $\lambda=\omega h\sqrt{\frac{\rho_2}{Q^2_{66}}}$ where ω is natural frequency.

Reddy and Kuppusamy[20] also presented a finite element formulation using the Srinivas' theory and solved free vibration problem of three layered simply supported laminates whose each layer is isotropic and has the same material properties with Poisson's ratio $\nu=0.3$. The code developed herein was also applied to Reddy's problem. The non-dimensionalized fundamental frequencies obtained by Reddy and Kuppusamy and the present analysis are shown in Table 3 along with elasticity solution. Apparently, the present analysis yields much more accurate results than Reddy's, even with a coarse mesh.

Table 3. Comparison of Reddy's and Present FEM Solution for Non-dimensionalized Natural Frequency λ of Isotropic Laminate

	Exact Sol. (Srinivas)	Reddy's Sol. (4×4 mesh)	Present Results 2×2 mesh	Present Results 4×4 mesh
λ	0.09315	0.0963	0.0951	0.09319
Error	–	3.3%	2.2%	0.04%

These numerical tests clearly show that the present finite element formulation based on the discrete laminate theory, which incorporates the transverse shear deformation in consistent manner, proves to be more accurate than earlier ones, e.g. Srinivas[21], Sun[31], etc., in predicting natural frequencies of laminated composite plates. Although further examination of the performance of the present model for other types of laminated plates is not possible since elasticity solutions available are limited, it is apparent that it is expected to yield better accuracy in other problems. In consequence, use of the presented procedure appears to be desirable to ensure general reliability of vibration analysis.

5. CONCLUSION

Using a variational principle derived in previous paper, which is based upon a consistent shear deformable discrete laminate theory, a displacement finite element formulation for the dynamic analysis of laminated plates has been presented and tested. As the nodal field variables, in-plane displacement of a layer, rotations of the cross sections of the remaining layers and transverse displacement of the plate were taken. This enables one the specify displacement boundary conditions layerwisely along plate edge surface. Performance of the formulation presented herein was tested and verified for the sandwitch-type laminated composite plate whose elasticity solution is known. Although the numerical test has been done for limited cases, it was apparent to yield excellent accuracy. Comparison of the test results also proved that the present formulation is better than other models. It is expected that the advantage of the developed code can be more pronounced in multilayered hybrid laminates, in which the case the effect of transverse shear deformation appears significantly.

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APPENDIX A :

Matrices and vectors in the spatially discretized functional (4) are defined as below. Herein, A^k, B^k, D^k are, respectively, the matrix form of transverse shear stiffness $A^k_{\alpha\beta\gamma\delta}$, $B^k_{\alpha\beta\gamma\delta}$ and $D^k_{\alpha\beta\gamma\delta}$, given in Eq.(22) of the companion paper[28], which represent the shear coupling between layers and are determined by layer material properties, fiber orientations, etc. [27].

$$M^k_{uu} = \int_{R^k_0} H_u P^k H_u^T\, dR \qquad K^{kj}_{ww} = \int_{R^k_0} T_s \Lambda^k T_s^T\, dR \qquad (p^0, p^n) = \int_{R^k_0} H_w (T_3^0, T_3^n)\, dR$$

$$M^k_{u\phi} = \int_{R^k_0} H_u P^k H_\phi^T\, dR$$

$$M^k_{\phi\phi} = \int_{R^k_0} H_\phi^T P^k H_\phi^T\, dR \qquad K^{kj}_{w\phi} = \int_{R^k_0} T_s \Lambda^k H_\phi^T\, dR \qquad R^k_{n1} = \int_{S^k_{1\sigma}} H_u n^k\, ds$$

$$M^k_{uuR} = \int_{R^k_0} H_u R^k H_u^T\, dR \qquad K^{kj}_{\phi\phi} = \int_{R^k_0} H_\phi \Lambda^k H_\phi^T\, dR \qquad R^k_{n2} = \int_{S^k_{1\sigma}} H_\phi n^k\, ds$$

$$M^k_{u\phi R} = \int_{R^k} H_u R^k H_\phi^T\, dR \qquad f^k = \int_{R^k_0} H_u F^k\, dR \qquad R^k_m = \int_{S^k_{2\sigma}} H_\phi m^k\, ds$$

$$M^k_{\phi\phi I} = \int_{R^k_e} H_\phi I^k H^T_\phi \, dR \qquad f^k_3 = \int_{R^k_e} H_w F^k_3 \, dR \qquad R^k_q = \int_{S^k_w} H_w q^k \, ds$$

$$M^k_{\phi\phi R} = \int_{R^k_e} H_\phi R^k H^T_\phi \, dR \qquad g^k = \int_{R^k_e} H_\phi G \, dR \qquad X^k = [X^k_1, X^k_2]^T$$

$$Y^k = [Y^k_1, Y^k_2]^T$$

$$M^k_{ww} = \int_{R^k_e} H_w P^k H^T_w \, dR \qquad h^k = \int_{R^k_e} H_\phi F^k \, dR \qquad F^k = [F^k_1, F^k_2]^T$$

$$K^k_{uuA} = \int_{R^k_e} T_u A^k T^T_u \, dR \qquad r^k_x = \int_{R^k_e} H_\phi X^k \, dR \qquad G^k = [G^k_1, G^k_2]^T$$

$$T^o = [T^o_1, T^o_2]^T$$

$$K^k_{u\phi A} = \int_{R^k_e} T_u A^k T^T_\phi \, dR \qquad b^k_x = \int_{R^k_e} H_u X^k \, dR \qquad T^n = [T^n_1, T^n_2]^T$$

$$K^k_{u\phi B} = \int_{R^k_e} T_u B^k T^T_\phi \, dR \qquad b^k_y = \int_{R^k_e} H_\phi Y^k \, dR \qquad n^k = [\hat{N}^k_1, \hat{N}^k_2]$$

$$m^k = [\hat{M}^k_1, \hat{M}^k_2]$$

$$K^k_{\phi\phi A} = \int_{R^k_e} T_\phi A^k T^T_\phi \, dR \qquad b^k_z = \int_{R^k_e} H_w Z^k \, dR \qquad q^k = [\hat{Q}^k_1, \hat{Q}^k_2]$$

$$K^k_{\phi\phi B} = \int_{R^k_e} T_\phi B^k T^T_\phi \, dR \qquad t^n = \int_{R^k_e} H_\phi T^n \, dR$$

$$K^k_{\phi\phi D} = \int_{R^k_e} T_\phi D^k T^T_\phi \, dR \qquad (\tau^o, \tau^n) = \int_{R^k_e} H_u (T^o, T^n) \, dR$$

APPENDIX B :

In the semidiscretized functional (5), elements of the load vector and system matrix are given as below.

$$S_{11} = \sum_{k=1}^{n} (M^k_{uu} + t^e K^k_{uuA})$$

$$(S_{12})_{1j} = t_j \sum_{k=j+1}^{n} (M^k_{u\phi\rho} + t^e K^k_{u\phi A}) + M^j_{u\phi R} + t^e K^j_{u\phi B} \; , \; j = 1, 2, \ldots, n-1$$

$$(S_{12})_{1n} = M^n_{u\phi R} + t^e K^n_{u\phi B}$$

$$(S_{22})_{ii} = t_i^2 \sum_{k=i+1}^{n} (M^k_{\phi\phi\rho} + t^e K^k_{\phi\phi A}) + M^i_{\phi\phi I} + t^e (K^i_{\phi\phi D} + K^{ii}_{s\phi\phi}) \; , \quad i = 1, 2, \ldots, n-1$$

$$(S_{22})_{nn} = M^n_{\phi\phi I} + t^e (K^n_{\phi\phi D} + K^{nn}_{s\phi\phi})$$

$$(S_{22})_{ij} = t_i t_j \sum_{k=j+1}^{n} (M^k_{\phi\phi\rho} + t^e K^k_{\phi\phi A}) + t_i M^j_{\phi\phi R} + t^e (t_i K^j_{\phi\phi B} + K^{ij}_{s\phi\phi})$$

$$i = 1, 2, \ldots, n-1 \text{ and } j = i+1, i+2, \ldots, n-1$$

$$(S_{22})_{in} = t_i M^n_{\phi\phi R} + t^e (t_i K^n_{\phi\phi B} + K^{in}_{s\phi\phi}), \quad i = 1, 2, \ldots, n-1$$

$$(S_{23})_{i1} = t^e \sum_{k=1}^{n} K^{ki}_{sw\phi} \quad i = 1, 2, \ldots, n-1$$

$$S_{33} = \sum_{k=1}^{n} \left[M^k_{ww} + t^e \sum_{j=1}^{n} K^{kj}_{sww} \right]$$

$$P_x = \sum_{k=1}^{n} \left(t^e (f^k - R^k_{n1}) + b^k_x \right) + t^e (\tau^n - \tau^o)$$

$$P_{\phi^k} = \sum_{i=i+k}^{n} \left(t^e (t_k h^i - R^i_{nx}) + t_k r^i_x \right) + t^e (g^k + t_k \tau^n - R^k_m) + b^k_y \; , \quad k = 1, 2, \ldots, n-1$$

$$P_{\phi^n} = t^e (g^n + t_n \tau^n - R^n_m) + b^n_y$$

$$P_w = \sum_{k=1}^{n} \left(t^e (f^k_3 - R^k_q) + b^k_z - p^o + p^n \right)$$