복합재료적층판의 진동해석을 위한 유한요소모델 I. 변분원리의 유도

Finite Element Analysis for Vibration of Laminated Plate Using a Consistent Discrete Theory Part I : Variational Principles

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요 약

적층판의 동적거동에 대한 유한요소해석모델개발을 목적으로 전단변형을 적합하게 고려한 적층판이론에 대한 변분원리를 유도하였다. 유도방법은 Sandhu 등에 의해 개발된 다변수 경계치문제의 변분원리이론을 따랐으며, 지배방정식의 미분연산자 매트릭스를 self-adjoint로 만들기 위하여 convolution을 이중선형사상으로 사용하였다. 유도된 적층판의 범함수에는 경계조건, 초기조건뿐만 아니라 유한요소해석모델에서 생길 수 있는 요소간 불연속조건도 포함시킬 수 있다. 상태변수의 적합함수공간을 확장하거나 특정조건을 적용하므로서 다양한 형태의 범함수를 유도할 수 있으며, 이를 통해 다양한 유한요소해석모델의 개발이 가능함을 논하였다.

Abstract

A family of variational principles governing the dynamics of laminated plate has been derived using a variationally consistent shear deformable discrete laminated plate theory with particular reference to finite element procedures. The theoretical basis for the derivation is Sandhu's generalized procedure for the variational formulation of linear coupled boundary value problem. As the bilinear mapping to write the operator matrix of the field equations in self-adjoint form, convolution product was employed. Boundary conditions, initial conditions and probable internal discontinuity were explicitly included in the governing functionals. Some interesting extensions and specializations of the general variational principle were presented, which can provide many different finite element formulations for the problem.

1. Introduction

For composite laminates, the governing equations are quite complicated due to material inhomogeneity and anisotropy. This makes analytical solutions difficult to obtain. Some solutions

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이 논문에 대한 토론을 1995년 6월 30일까지 본 학회에 보내주시면 1995년 12월호에 그 결과를 게재하겠습니다.

of the complete elasticity equations or plate equations have been obtained for problems with simple lamination scheme and geometry. However, for laminated plates with arbitrary stacking sequence, irregular geometry and mixed boundary conditions, the problem is intractable and use of some approximation technique such as finite element method is inevitable. Finite element procedures for obtaining approximate solution of boundary value problems are often based on variational formulations. For systematic development of variational principles of the boundary value problem, many researches have been done[1-8] and most recently Sandhu et al.[5-8] established general framework for the coupled boundary value problem of multivariables by extending Mikhlin's basic theorem[1]. A coupled boundary value problem can be stated as

$$A_{ij}u_j=f_i \qquad \text{on } R_i \tag{1}$$

$$C_{ij}u_j=g_i \qquad \text{on } \partial R_i \tag{2}$$

where A_{ij}, C_{ij} are the linear, bounded operators defined over the region R_i and on its boundary ∂R_i : u_i are the field variables ; f_i, g_i are the given functions on R_i and ∂R_i, respectively. For this problem, variational formulation is stated as

$$\Omega=\sum_{i=1}^{n}\sum_{j=1}^{n}\{<u_i,\ A_{ij}u_j-2f_i>_R+<u_i,\ C_{ij}u_j-2g_i>_{\partial R}\} \tag{3}$$

if the operators A_{ij} are self-adjoint with respect to a certain bilinear mapping $<,>$ and the boundary operators C_{ij} are consistent with A_{ij}, i.e.

$$\sum_{j=1}^{n}<v_j,\ A_{ji}u_i>_R=<u_i,\ \sum_{j=1}^{n}A_{ij}v_j>_R+<v_i,\ \sum_{j=1}^{n}C_{ij}u_j>_{\partial R}-\sum_{j=1}^{n}<u_j,\ C_{ij}v_i>_{\partial R} \tag{4}$$

where a subscript associated with bilinear mapping indicates the domain of definition. This implies that Gateaux differential of the functional Ω in (3) vanishes if Eqs. (1) and (2) are satisfied. Inverse statement is valid, i.e., vanishment of Gateaux differential of Ω means that (1) and (2) are satisfied. In connection with application to the finite element procedure, it may be necessary to allow for discontinuities of certain field variables along interelement boundaries since some quantities may have limited continuity in approximation space, e.g., when non-conforming element is used. Such discontinuity can be included in the variational formulation by expressing it in the same form as the boundary conditions. In consequence, a variational formulation of the boundary value problem is to find self-adjoint form of the field equations with respect to certain bilinear mapping and boundary conditions consistent with the field operators. Recently, Al-Gothani[12], following the procedure stated previously, presented a complementary variational formulation of dynamics of laminated composite plate using the field equations of displacement-based discrete laminate theory[9-11]. Various extended and specialized forms of the general variational principles were discussed. However, there is another way to derive variational principles,

i.e., direct variational formulation which gives other types of variational principles. Furthermore, the laminate theory used does not treat the effect of transverse shear deformation properly which is important in the analysis of composite laminates. In this paper, we present variational formulations of the dynamics of laminated plate allowing for nonhomogeneous boundary conditions as well as internal discontinuities using a discrete laminated plate theory that incorporates the effect of transverse shear deformation in variationally consistent manner[13]. Both the direct and the complementary formulations are considered along with extensions and specialization of general variational principles.

2. A Consistent Shear Deformable Discrete Laminate Theory

Differential Form of Field Equations

Consider a laminated plate with uniform thickness h, which is composed of arbitrary number of thin layers and enclosed by a cylindrical surface S along its edge and two parallel planes R(Fig. 1). Each layer is homogeneous and orthotropic with its material axes not necessarily along with geometric coordinate axes. All the layers are assumed perfectly bonded together. Using Cartesian reference frame, global coordinate axes are defined in a way that x_α axes are on the bottom surface of the plate and x_3 axis is normal to this plane. In addition, local coordinate system, x^k_i, is defined for the k^{th} layer, but the range of x^k_3 is limited to the thickness of the layer. In the following discussion, standard indicial notation is used. Summation on repeated indices is implied, where the Latin indices take on the range of 1, 2, 3 and Greek indices take on 1 and 2(Fig. 2).

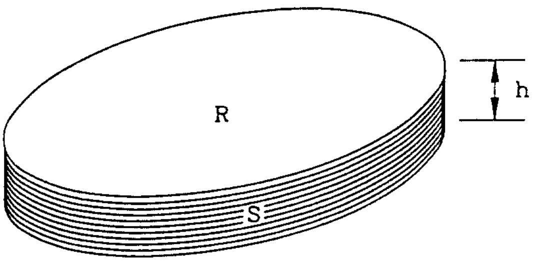

Fig. 1. Laminated Plate

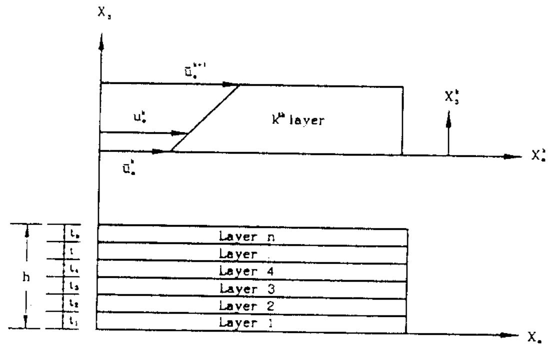

Fig. 2. Global and Local Coordinate Systems in a Laminated Plate

The displacements are assumed over the thickness of k^{th} layer as

$$u^k_\alpha(x_i, t)=\bar{u}^k_\alpha(x_\beta, t)+x^k_3\,\phi^k_\alpha(x_\beta, t) \tag{5}$$

$$u^k_3(x_i, t)=w^k(x_\beta, t) \tag{6}$$

where u^k_i are the components of the displacement vector ; $\bar{u}^k_\alpha$, w^k are the associated displacements at the bottom surface of the k^{th} layer ; ϕ^k_α are the rotations of a cross section of the k^{th} layer ; t is time. For small deformation, the kinematic relations with (5) and (6) are

$$e^k_{\alpha\beta}=\bar{e}^k_{\alpha\beta}+x^k_3\,\kappa^k_{\alpha\beta},\quad e^k_{33}=0\quad e^k_{\alpha3}=\frac{1}{2}(\phi^k_\alpha+w_\alpha) \tag{7}$$

where a comma indicates the partial differentiation with respect to the coordinates and

$$\bar{e}^k_{\alpha\beta}=\frac{1}{2}(\bar{u}^k_{\alpha,\beta}+\bar{u}^k_{\beta,\alpha})\qquad \kappa^k_{\alpha\beta}=\frac{1}{2}(\phi^k_{\alpha,\beta}+\phi^k_{\beta,\alpha}) \tag{8}$$

For the k^{th} layer, the dynamic plate equilibrium equations are given by

$$\begin{aligned} &N^k_{\alpha\beta,\beta}+F^k_\alpha+(T^k_\alpha-T^{k-1}_\alpha)-P^k\ddot{u}^k_\alpha-R^k\ddot{\phi}^k_\alpha=0\\ &Q^k_{\alpha,\alpha}+F^k_3+(T^k_3-T^{k-1}_3)-P^k_3\ddot{w}^k=0\\ &M^k_{\alpha\beta,\beta}-Q^k_\alpha+G^k_\alpha+t_kT^k_\alpha-R^k\ddot{u}^k_\alpha-I^k\ddot{\phi}^k_\alpha=0 \end{aligned} \tag{9}$$

where superposed dots indicate time derivatives and

$$\begin{aligned} &Q^k_\alpha=\int_0^{t_k}\sigma^k_{\alpha3}\,dx^k_3\quad F^k_3=\int_0^{t_k}f^k_3\,dx^k_3\quad (N^k_{\alpha\beta},M^k_{\alpha\beta})=\int_0^{t_k}(1,x^k_3)\sigma^k_{\alpha\beta}\,dx^k_3\\ &(F^k_\alpha,G^k_\alpha)=\int_0^{t_k}(1,x^k_3)\,f^k_3\,dx^k_3\quad \{P^k,R^k,I^k\}=\int_0^{t_k}\{1,x^k_3,(x^k_3)^2\}\,\rho^k\,dx^k_3 \end{aligned} \tag{10}$$

Here, T^k_i and T^{k-1}_i denote the stres components of σ^k_{i3} on the top and bottom surfaces of the k^{th} layer, respectively ; f_i is the components of body force vector ; and t_k and ρ^k are the thickness and mass density of the k^{th} layer.

For the k^{th} layer with a material symmetry about its middle plane, the plate constitutive equations are obtained using stress-strain relations of elasticity in the definitions of stress resultants.

$$\begin{Bmatrix} N^k_{\alpha\beta}\\ M^k_{\alpha\beta}\end{Bmatrix}=\begin{bmatrix} A^k_{ij} & B^k_{ij}\\ B^k_{ij} & D^k_{ij}\end{bmatrix}\begin{Bmatrix}\bar{e}^k_{\alpha\beta}\\ \kappa^k_{\alpha\beta}\end{Bmatrix}\qquad Q^k_\alpha=2\sum_{j=1}^{n}\lambda^{kj}_{\alpha\beta}\,e^k_{\beta3} \tag{11}$$

where

$$[A^k_{ij},B^k_{ij},D^k_{ij}]=\int_0^{t_k}[1,x^k_3,(x^k_3)^2]\bar{Q}^k_{ij}\,dx^k_3 \qquad i,j=1,2,6 \tag{12}$$

Here, $\bar{Q}^k_{ij}$ are reduecd elastic constants obtained by using generalized plane stress state and $\lambda^{kj}_{\alpha\beta}$ are coefficients to be determined by material properties, thickness and stacking sequence of a laminate. The coupled transverse shear constitutive equations in (11) derived by Hong[13] account for the parabolic distribution over the thickness of a layer and interlayer continulity of transverse shear stresses, Equivalently, the inverse relations of (11) can be stated as

$$\begin{Bmatrix} \bar{e}^k_{\alpha\beta} \\ \kappa^k_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{A}^k_{\alpha\beta\gamma\delta} & \bar{B}^k_{\alpha\beta\gamma\delta} \\ \bar{B}^k_{\alpha\beta\gamma\delta} & \bar{D}^k_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{Bmatrix} N^k_{\gamma\delta} \\ M^k_{\gamma\delta} \end{Bmatrix} \quad (\phi^k_\alpha + w^k_{,\alpha}) = \sum_{j=1}^{n} \mu^{kj}_{\alpha\beta} Q^j_\beta \tag{13}$$

For these field equations, the following boundary and initial conditions need to be specified.

$$\begin{aligned} &N^k_{\alpha\beta}\eta_\beta = \hat{N}^k_\alpha(x_\beta, t) && \text{on } S^k_1 x[0, \infty), && M^k_{\alpha\beta}\eta_\beta = \hat{M}^k_\alpha(x_\beta, t) && \text{on } S^k_3 x[0, \infty) \\ &Q^k_\alpha = \hat{Q}^k_\alpha(x_\beta, t) && \text{on } S^k_5 x[0, \infty), && \bar{u}^k_\alpha = \hat{\bar{u}}^k_\alpha(x_\beta, t) && \text{on } S^k_2 x[0, \infty) \\ &\phi^k_\alpha = \hat{\phi}^k_\alpha(x_\beta, t) && \text{on } S^k_4 x[0, \infty), && u^k = \hat{w}^k(x_\beta, t) && \text{on } S^k_6 x[0, \infty) \end{aligned} \tag{14}$$

$$\begin{aligned} &\bar{u}^k_\alpha(x_\beta, 0) = \bar{u}^k_{\alpha o}(x_\beta), && \phi^k_\alpha(x_\beta, 0) = \phi^k_{\alpha o}(x_\beta), && w^k(x_\beta, 0) = w^k_o(x_\beta) \\ &\dot{\bar{u}}^k_\alpha(x_\beta, 0) = \dot{\bar{u}}^k_{\alpha o}(x_\beta), && \dot{\phi}^k_\alpha(x_\beta, 0) = \dot{\phi}^k_{\alpha o}(x_\beta), && \dot{w}^k(x_\beta, 0) = \dot{w}^k_o(x_\beta) \end{aligned} \tag{15}$$

where a circumflex denotes the value of the prescribed quantity on S^k ; η_β are components of the unit outward normal to S^k ; $[0, \infty)$ is the positive time interval. The boundary segments S^k_1, S^k_2 are complementary subsets of S^k and so are S^k_3, S^k_4 and S^k_5, S^k_6. Since the all the layers are assumed to be perfectly bonded, the following continuity conditions of the displacements and stresses in the layer interfaces are needed to complete laminate theory.

$$\bar{u}^{k+1}_\alpha = \bar{u}^k_\alpha + t_k\phi^k_\alpha \text{ and } w^{k+1} = w^k \tag{16}$$

$$\sigma^k_{i3}(x^k_3 = t_k) = \sigma^{k+1}_{i3}(x^{k+1}_3 = 0) \tag{17}$$

Through these continuity conditions, all the field equations defined for each layer are combined to give the field equations of a laminate. Here, it should be noted that the stress continuities have been already enforced in the equilibrium equations (9) by T^k_i.

<u>Integral Form of Field Equations</u>

To set up variational pinciple for laminated plates, it is necessary to write the field equations in a way that the operator matrix is self-adjoint in a certain space. The self-adjointness of operators is not an absolute notion, but rather, it is relative to a bilinear mapping. Thus, there are two possible ways to set up variational formulation of the problem ; one way is to find a bilinear mapping that makes the field operators be self-adjoint and another way is to rewrite the field equations in different form so that they can be self-adjoint with respect to a familiar bilinear mapping. For the present problem, we follow the latter way following Gurtin's procedure although other ways may be chosen. The differential form of field equations are transformed to the equiavalent integral form by applying Laplace Transform and taking inverse after appropriate rearrangement. The procedure removes the time derivatives from the equilibrium equations and includes initial conditions explicitly. For the field equations and boundary conditions given previously, the equivalent integral form is

$$\text{Kinematics : } t * e^k_{\alpha\beta} = t * \bar{e}^k_{\alpha\beta} + x^k_3 t * \kappa^k_{\alpha\beta} \qquad t * e^k_{\alpha 3} = \frac{1}{2} t * (\phi^k_\alpha + w^k_{,\alpha}) \tag{18}$$

where

$$t * \bar{e}^k_{\alpha\beta} = \frac{1}{2} t * (\bar{u}^k_{\alpha,\beta} + \bar{u}^k_{\beta,\alpha}) \quad t * \kappa^k_{\alpha\beta} = \frac{1}{2} t * (\phi^k_{\alpha,\beta} + \phi^k_{\beta,\alpha}) \tag{19}$$

Equilibrium :

$$\begin{aligned} & t * N^k_{\alpha\beta,\beta} + t * (T^k_\alpha - T^{k-1}_\alpha) + t * F_\alpha - P^k \bar{u}^k_\alpha - R^k \phi^k_\alpha + X^k_\alpha = 0 \\ & t * M^k_{\alpha\beta,\beta} - t * Q_\alpha + t * (t_k\ T^k_\alpha + G_\alpha) - R^k \bar{u}^k_\alpha - I^k \phi^k_\alpha + Y^k_\alpha = 0 \\ & t * Q^k_{\alpha,\alpha} + t * (T^k_3 - T^{k-1}_3) + t * F^k_3 - P^k w^k + Z^k = 0 \end{aligned} \tag{20}$$

where

$$\begin{aligned} & X^k_\alpha = P^k (\bar{u}^k_{\alpha 0} + t * \dot{\bar{u}}^k_{\alpha 0}) + R^k (\phi^k_{\alpha 0} + t * \dot{\phi}^k_{\alpha 0}) \\ & Y^k_\alpha = R^k (\bar{u}^k_{\alpha 0} + t * \dot{\bar{u}}^k_{\alpha 0}) + I^k (\phi^k_{\alpha 0} + t * \dot{\phi}^k_{\alpha 0}) \\ & Z^k = P^k (w^k_0 + t * \dot{w}^k_0) \end{aligned} \tag{21}$$

Constitutive Relations :

$$t * \begin{bmatrix} N^k_{\alpha\beta} \\ M^k_{\alpha\beta} \end{bmatrix} = t * \begin{bmatrix} A^k_{\alpha\beta\gamma\delta} & B^k_{\alpha\beta\gamma\delta} \\ B^k_{\alpha\beta\gamma\delta} & D^k_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} \bar{e}^k_{\gamma\delta} \\ k^k_{\gamma\delta} \end{bmatrix} \qquad t * \begin{bmatrix} \bar{e}^k_{\alpha\beta} \\ \kappa^k_{\alpha\beta} \end{bmatrix} = t * \begin{bmatrix} \bar{A}^k_{\alpha\beta\gamma\delta} & \bar{B}^k_{\alpha\beta\gamma\delta} \\ \bar{B}^k_{\alpha\beta\gamma\delta} & \bar{D}^k_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} N^k_{\gamma\delta} \\ M^k_{\gamma\delta} \end{bmatrix}$$

$$t * Q^k_\alpha = 2t * \sum_{j=1}^{n} \lambda^{kj}_{\alpha\beta} e^j_{\beta 3} \qquad t * (\phi^k_\alpha + w^k_{,\alpha}) = t * \sum_{j=1}^{n} \mu^{kj}_{\alpha\beta} Q^j_\beta \tag{22}$$

Interlayer Continuity of Displacements :

$$t * \bar{u}^{k+1}_\alpha = t * \bar{u}^k_\alpha + t * t_k \phi^k_\alpha \qquad t * w^k = t * w^{k+1} \tag{23}$$

Boundary Conditions :

$$\begin{aligned} & t * N^k_{\alpha\beta}\, \eta_\beta = t * \hat{N}_\alpha \text{ on } S^k_1, \quad && t * M^k_{\alpha\beta}\, \eta_\beta = t * \hat{M}_\alpha \text{ on } S^k_3 \\ & t * Q^k_\alpha = t * \hat{Q}^k_\alpha \quad \text{on } S^k_5, && t * \bar{u}^k_\alpha = t * \hat{\bar{u}}^k_\alpha \quad \text{on } S^k_2 \\ & t * \phi^k_\alpha = t * \hat{\phi}^k_\alpha \quad \text{on } S^k_4, && t * w^k = t * \hat{w}^k \quad \text{on } S^k_6 \end{aligned}$$

Here, an asterisk(*) denotes convolution product

$$u * v = \int_0^t u(\tau) v(t - \tau) d\tau \tag{25}$$

3. Direct Variational Formulation

General Variational Principle

For a laminate composed of n layers, the field equation (18)−(23) in integral form can be written in the self-adjoint matrix form as

$$\begin{bmatrix} A_1 & B_1 & D_{1,2} & 0 & D_{1,3} & 0 & \cdots & D_{1,n-1} & 0 & D_{1,n} \\ & 0 & C^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & A_2 & B_2 & D_{2,3} & 0 & \cdots & D_{2,n-1} & 0 & D_{2,n} \\ & & & 0 & C^T & 0 & \cdots & 0 & 0 & 0 \\ & & & & A_3 & B_3 & \cdots & D_{3,n-1} & 0 & D_{3,n} \\ & & & & & 0 & & 0 & 0 & 0 \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & A_{n-1} & B_{n-1} & D_{n-1,n} \\ & & & & & & & & 0 & C^T \\ & & & & & & & & & A_n \end{bmatrix} \begin{Bmatrix} U_1 \\ \Xi_1 \\ U_2 \\ \Xi_2 \\ U_3 \\ \Xi_3 \\ \cdot \\ \cdot \\ U_{n-1} \\ \Xi_{n-1} \\ U_n \end{Bmatrix} = \begin{Bmatrix} -r_1 + \Xi_0 \\ 0 \\ -r_2 \\ 0 \\ -r_3 \\ 0 \\ \cdot \\ \cdot \\ -r_{n-1} \\ 0 \\ -r_n - \Xi_n \end{Bmatrix} \tag{26}$$

where only the operators in the upper triangular region have been entered. The operators below diagonal are adjoints of the operators above diagonal, i.e., the operator A_{ij} is adjoint of A_{ji} in the sense of the bilinear mapping used to set up the variational formulation, i.e.,

$$\langle u, v \rangle_R = \int_R u^* v \, dv \tag{27}$$

which is linear and nondegenerate[2,3]. Explicitly, the symbolic operators applying in (26) are :

$$A_k = \begin{bmatrix} -P^k\delta_{\alpha\gamma} & 0 & t^*\Gamma_1 & -R^k\delta_{\alpha\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & -t^*A^k_{\alpha\beta\gamma\delta} & t^* & 0 & -t^*B^k_{\alpha\beta\gamma\delta} & 0 & 0 & 0 & 0 \\ -t^*\Gamma_2 & t^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -R^k\delta_{\alpha\gamma} & 0 & 0 & -I^k\delta_{\alpha\gamma} & 0 & t^*\Gamma_1 & 0 & 0 & -t^* \\ 0 & -t^*B^k_{\alpha\beta\gamma\delta} & 0 & 0 & -t^*D^k_{\alpha\beta\gamma\delta} & t^* & 0 & 0 & 0 \\ 0 & 0 & 0 & -t^*\Gamma_2 & t^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -P^k & 0 & t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^*\lambda^{kk}_{\gamma\alpha} & t^* \\ 0 & 0 & 0 & -t^* & 0 & 0 & -t^*\delta_{\alpha\gamma}\dfrac{\partial}{\partial\gamma} & t^* & 0 \end{bmatrix}$$

$$B_k^T = \begin{bmatrix} t^* & 0 & 0 & t^* t_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^* & 0 & 0 \end{bmatrix} \qquad C_k^T = \begin{bmatrix} -t^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -t^* & 0 & 0 \end{bmatrix}$$

$$D_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0\;0\;0\;0 & 0 & 0 \\ & 0 & 0 & 0\;0\;0\;0 & 0 & 0 \\ & & 0 & 0\;0\;0\;0 & 0 & 0 \\ & & & 0\;0\;0\;0 & 0 & 0 \\ & & & 0\;0\;0 & 0 & 0 \\ & & & 0\;0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{symm.} & & & -t^* \lambda_{\gamma\alpha}^{ij} & 0 \\ & & & & & 0 \end{bmatrix} \qquad i,j = 1, 2, \ldots, n$$

$$U_k^T = [\bar{u}_\alpha^k, \bar{e}_{\gamma\delta}^k, N_{\alpha\beta}^k, \phi_\alpha^k, \kappa_{\gamma\delta}^k, M_{\alpha\beta}^k, w^k, 2e_{\alpha 3}^k, Q_\gamma^k]$$

$$\Xi_k^T = [T_\alpha^k, T_3^k]$$

$$r_k^T = [t^* F_\gamma^k + X_\gamma^k, 0, 0, t^* G_\gamma + Y_\gamma^k, 0, 0, t^* F_3^k + Z^k, 0, 0]$$

$$\Xi_0^T = [t^* T_\alpha^0, 0, 0, 0, 0, 0, t^* T_3^0, 0, 0]$$

$$\Xi_n^T = [t^* T_\alpha^n, 0, 0, t^* t_n T_\alpha^n, 0, 0, t^* T_3^n, 0, 0]$$

$$\Gamma_1 = \frac{1}{2}\left(\delta_{\alpha\gamma} \frac{\partial}{\partial \beta} + \delta_{\beta\gamma} \frac{\partial}{\partial \alpha}\right), \qquad \Gamma_2 = \frac{1}{2}\left(\delta_{\alpha\gamma} \frac{\partial}{\partial \delta} + \delta_{\alpha\delta} \frac{\partial}{\partial \gamma}\right)$$

Here, T_i^0 and T_i^n are specified shear stress components on the top and bottom surface of the plate and $\delta_{\alpha\beta}$ is Kronecker's delta. The boundary conditions consistent with the operator equations (26) are

$$\begin{aligned} t^* N_{\alpha\beta}^k \eta_\beta &= t^* \hat{N}_\alpha \quad \text{on } S_1^k(x_\alpha^k) & t^* M_{\alpha\beta}^k \eta_\beta &= t^* \hat{M}_\alpha \quad \text{on } S_3^k(x_\alpha^k) \\ t^* Q_\alpha^k \eta_\alpha &= t^* \hat{Q}_\alpha^k \eta_\alpha \quad \text{on } S_5^k(x_\alpha^k) & t^* \bar{u}_\alpha^k \eta_\beta &= t^* \hat{\bar{u}}_\beta^k \eta_\alpha \quad \text{on } S_2^k(x_\alpha^k) \\ t^* \phi_\alpha^k \eta_\beta &= t^* \hat{\phi}_\alpha^k \eta_\beta \quad \text{on } S_4^k(x_\alpha^k) & t^* w^k \eta_\alpha &= t^* \hat{w}^k \eta_\alpha \quad \text{on } S_6^k(x_\alpha^k) \end{aligned} \tag{28}$$

and internal jump discontinuity conditions are :

$$t^*(N^k_{\alpha\beta}\eta_\beta)' = t^*(g^k_1)_\alpha \quad \text{on } S^k_{1i}(x^k_\alpha) \quad t^*(M^k_{\alpha\beta}\eta_\beta)' = t^*(g^k_3)_\alpha \quad \text{on } S^k_{3i}(x^k_\alpha)$$

$$t^*(Q^k_\alpha\eta_\alpha)' = t^*(g^k_5)_\alpha\eta_\alpha \quad \text{on } S^k_{5i}(x^k_\alpha) \quad t^*(\bar{u}^k_\alpha\eta_\beta)' = t^*(g^k_2)_\alpha\eta_\beta \quad \text{on } S^k_{2i}(x^k_\alpha) \tag{29}$$

$$t^*(\phi^k_\alpha\eta_\beta)' = t^*(g^k_4)_\alpha\eta_\beta \quad \text{on } S^k_{4i}(x^k_\alpha) \quad t^*(w^k\eta_\alpha)' = t^*(g^k_6)\eta_\alpha \quad \text{on } S^k_{6i}(x^k_\alpha)$$

By the definitions (4), following relations are satisfied between off-diagonal operators.

$$\begin{aligned} <\bar{u}^k_\alpha, t^*N^k_{\alpha\beta,\beta}>_{R^k} = &-<t^*\bar{u}^k_{\alpha,\beta}, N^k_{\alpha\beta}>_{R^k} - <\bar{u}^k_\alpha, t^*N^k_{\alpha\beta}\eta_\beta>_{S^k_1} + <N^k_{\alpha\beta}, t^*\bar{u}^k_\alpha\eta_\beta>_{S^k_2} \\ &- <\bar{u}^k_\alpha, t^*(N^k_{\alpha\beta}\eta_\beta)'>_{S^k_{1i}} + <N^k_{\alpha\beta}, t^*(\bar{u}^k_\alpha\eta_\beta)'>_{S^k_{2i}} \end{aligned} \tag{30}$$

$$\begin{aligned} <\phi^k_\alpha, t^*M^k_{\alpha\beta,\beta}>_{R^k} = &-<t^*\phi^k_{\alpha,\beta}, M^k_{\alpha\beta}>_{R^k} - <\phi^k_\alpha, t^*M^k_{\alpha\beta}\eta_\beta>_{S^k_3} + <M^k_{\alpha\beta}, t^*\phi^k_\alpha\eta_\beta>_{S^k_4} \\ &- <\phi^k_\alpha, t^*(M^k_{\alpha\beta}\eta_\beta)'>_{S^k_{3i}} + <M^k_{\alpha\beta}, t^*(\phi^k_\alpha\eta_\beta)'>_{S^k_{4i}} \end{aligned} \tag{31}$$

$$\begin{aligned} <w^k, t^*Q^k_{\alpha,\alpha}>_{R^k} = &-<t^*w^k_{,\alpha}, Q^k_\alpha>_{R^k} - <w^k, t^*Q^k_\alpha\eta_\alpha>_{S^k_5} + <Q^k_\alpha, t^*w^k\eta_\alpha>_{S^k_6} \\ &- <w^k, t^*(Q^k_\alpha\eta_\alpha)'>_{S^k_{5i}} + <Q^k_\alpha, t^*(w^k\eta_\alpha)'>_{S^k_{6i}} \end{aligned} \tag{32}$$

Thus, using the Eq. (3), the governing functional for the operator equations (26) is defined as

$$\begin{aligned} \Omega = &\sum_{k=1}^{n} <U^T_k, A_kU_k>_{R^k} + \sum_{k=1}^{n-1} <U^T_k, B_k\Xi_k>_{R^k} + \sum_{k=2}^{n} <U^T_k, C\Xi_{k-1}>_{R^k} \\ &+ \sum_{k=1}^{n}\sum_{j=1}^{n} <U^T_k, D_{j,k}U_j>_{R^k} - \sum_{k=1}^{n} <U^T_k, D_{k,k}U_k>_{R^k} + \sum_{k=1}^{n} <\Xi^T_k, B^T_kU_k>_{R^k} \\ &+ \sum_{k=1}^{n-1} <\Xi^T_k, C^TU_{k+1}>_{R^k} + 2\sum_{k=1}^{n} <U^T_k, r_k>_{R^k} - 2<U^T_1, \Xi_0>_{R^1} \\ &+ 2<U^T_n, \Xi_n>_{R^n} + \text{Boundary Terms} + \text{Internal Jump Terms} \end{aligned} \tag{33}$$

It is easily shown that the Gateaux differential of (33) vanishes if and only if all the field equations along with the boundary conditions and internal jump discontinuity conditions are satisfied. Thus, the functional (33) is a variational formulation governing the dynamics of laminated plate.

Extensions and Specializations

The relations (30)−(32) may be used to eliminate either of $N^k_{\alpha\beta,\beta}$ or $\bar{u}^k_{\alpha,\beta}$, either of $M^k_{\alpha\beta,\beta}$ or $\phi^k_{\alpha,\beta}$ and either $Q^k_{\alpha,\alpha}$ or $w^k_{,\alpha}$ from the governing functional Ω in (33). This means that the requirement of differentiability of those variables is relaxed, thereby extending the space of admissible states in the functional. In the context of finite element procedure, it is clear that the extension of the admissible space provides greater freedom in selection of approximation function. Also, such extension leads to numerous different formulations, e.g., elimination of derivatives of all the stress variables leads to displacement formulation, elimination of derivatives of all the kinematic variables leads to stress formulation and selective elimination of derivatives of stress and kinematic variables leads to so-called mixed formulation.

In addition, if the admissible state of variables is constrained to satisfy some field equations and / or boundary conditions, certain specialized forms of the variational principle are obtained. This procedure is used to reduce the number of field variables in the governing functional, which is advantageous in the finite element formulation if the constrained condition can be satisfied easily. Also, specialized functional with certain assumptions in the spatial and temporal variation of some variables can lead to approximate theories.

To illustrate the procedure of extension and specialization of the general governing functional, we consider Ω. Eliminating $N^k_{\alpha\beta,\beta}$ $M^k_{\alpha\beta,\beta}$ and $Q^k_{\alpha,\alpha}$ from it by using (30) − (32), we have

$$
\begin{aligned}
\Omega_1 = \sum_{k=1}^{n}\{ & -<\bar{u}^k_\alpha, P^k\bar{u}^k_\alpha>_{R^k} - <\phi^k_\alpha, I^k\phi^k_\alpha>_{R^k} - <w^k, P^k w^k>_{R^k} \\
& -<\bar{e}^k_{\alpha\beta}, t^* A^k_{\alpha\beta\gamma\delta}\bar{e}^k_{\gamma\delta}>_{R^k} - <\kappa^k_{\alpha\beta}, t^* D^k_{\alpha\beta\gamma\delta}\kappa^k_{\gamma\delta}>_{R^k} - 2<\bar{u}^k_\alpha, R^k\phi^k_\alpha>_{R^k} \\
& +2<N^k_{\alpha\beta}, t^*(\phi^k_\alpha - \bar{u}^k_{\alpha,\beta})>_{R^k} + 2<M^k_{\alpha\beta}, t^*(\kappa^k_{\alpha\beta} - \phi^k_{\alpha,\beta})>_{R^k} \\
& -2<Q^k_\alpha, t^*(2e^k_{\alpha 3} - \phi^k_\alpha - w_{,\alpha})>_{R^k} - 2<\bar{e}^k_{\alpha\beta}, t^* B^k_{\alpha\beta\gamma\delta}\kappa^k_{\gamma\delta}>_{R^k}\}
\end{aligned}
$$

$$
+\sum_{k=1}^{n}\sum_{j=1}^{n} <2e^k_{\alpha 3}, -2t^*\lambda^{kj}_{\alpha\beta}e^j_{\beta 3}>_{R^k}
$$

$$
+\sum_{k=1}^{n-1}\{<T^k_\alpha, t^*(\bar{u}^k_\alpha + t_k\phi^k_\alpha - \bar{u}^{k+1}_\alpha)>_{R^k} + <T^k_3, t^*(w^k - w^{k+1})>_{R^k}\}
$$

$$
+2\sum_{k=1}^{n}\{ <\bar{u}^k_\alpha, t^* F^k_\alpha + X^k_\alpha>_{R^k} + <\phi^k_\alpha, t^* G^k_\alpha + Y^k_\alpha>_{R^k} + <w^k, t^* F^k_3 + Z^k>_{R^k} \}
$$

$$
-2<\bar{u}^1_\alpha, t^* T^0_\alpha>_{R^1} - 2<w^1, t^* T^0_3>_{R^1}
$$

$$
+2<\bar{u}^n_\alpha, t^* T^n_\alpha>_{R^n} + 2<\phi^n_\alpha, t^* t_n T^n_\alpha>_{R^k} + 2<w^n, t^* T^n_3>_{R^n}
$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^k,\ t^*\hat{N}_\alpha^k>_{S_1^k}-2<\phi_\alpha^k,\ t^*\hat{M}_\alpha^k>_{S_3^k}$$

$$-2<w^k,\ t^*\hat{Q}_\alpha^k\eta_\alpha>_{S_5^k}+2<N_{\alpha\beta}^k,\ t^*(\bar{u}_\alpha^k\eta_\beta-\hat{\bar{u}}_\alpha^k\eta_\beta)>_{S_2^k}$$

$$+2<M_{\alpha\beta}^k,\ t^*(\phi_\alpha^k\eta_\beta-\hat{\phi}_\alpha^k\eta_\beta)>_{S_4^k}+2<Q_\alpha^k,\ t^*(w^k\eta_\alpha-\hat{w}^k\eta_\alpha)>_{S_6^k}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}_\alpha^k,\ t^*(g_1^k)_\alpha>_{S_{1i}^k}-2<\phi_\alpha^k,\ t^*(g_3^k)_\alpha>_{S_{3i}^k}$$

$$-2<w^k,\ t^*(g_5^k)_\alpha\eta_\alpha>_{S_{5i}^k}+<N_{\alpha\beta}^k,\ t^*((\bar{u}_\alpha^k\eta_\beta)'-2(g_2^k)_\alpha\eta_\beta)>_{S_{2i}^k}$$

$$+<M_{\alpha\beta}^k,\ t^*((\phi_\alpha^k\eta_\beta)'-2(g_4^k)_\alpha\eta_\beta)>_{S_{4i}^k}+<Q_\alpha^k,\ t^*((w^k\eta_\alpha)'-2(g_6^k)\eta_\alpha)>_{S_{6i}^k} \tag{34}$$

where all the stress resultants do not need differentable, so their admissible space is extended. If we specialized Ω_1 to identically satisfy the kinematic relations (19) and interface continuity conditions of displacements (23),

$$\Omega_2=-\sum_{k=1}^{n}\{<\bar{u}_\alpha^1,\ P^k\bar{u}_\alpha^1>_{R^k}+2<\bar{u}_\alpha^1,\ P^k\sum_{i=1}^{k-1}t_i\phi^i>_{R^k}+<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ P^k\sum_{j=1}^{k-1}t_j\phi_\alpha^j>_{R^k}$$

$$+<\phi_\alpha^k,\ I^k\phi_\alpha^k>_{R^k}+<w,\ P^kw>_{R^k}+2<\bar{u}_\alpha^1,\ R^k\phi_\alpha^k>_{R^k}$$

$$+2<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ R^k\phi_\alpha^k>_{R^k}+<\bar{e}_{\alpha\beta}^1,\ t^*A_{\alpha\beta\gamma\delta}^k\bar{e}_{\gamma\delta}^1>_{R^k}$$

$$+2<\bar{e}_{\alpha\beta}^1,\ t^*A_{\alpha\beta\gamma\delta}^k\sum_{i=1}^{k-1}t_i\kappa_{\gamma\delta}^i>_{R^k}+<\sum_{i=1}^{k-1}t_i\kappa_{\alpha\beta}^k,\ t^*A_{\alpha\beta\gamma\delta}^k\sum_{j=1}^{k-1}t_j\kappa_{\gamma\delta}^j>_{R^k}$$

$$+<\kappa_{\alpha\beta}^k,\ t^*D_{\alpha\beta\gamma\delta}^k\kappa_{\gamma\delta}^k>_{R^k}+2<\bar{e}_{\alpha\beta}^1,\ t^*B_{\alpha\beta\gamma\delta}^k\kappa_{\gamma\delta}^k>_{R^k}+2<\sum_{i=1}^{k-1}t_i\kappa_{\alpha\beta}^i,\ t^*B_{\alpha\beta\gamma\delta}^k\kappa_{\gamma\delta}^k>\}$$

$$-\sum_{k=1}^{n}\sum_{j=1}^{n}<2e_{\alpha3}^k,\ +2t^*\lambda_{\alpha\beta}^{kj}e_{\beta3}^j>_{R^k}$$

$$+2\sum_{k=1}^{n}\{<\bar{u}_\alpha^1,\ t^*F_\alpha^k+X_\alpha^k>_{R^k}+<\sum_{i=1}^{k-1}t_i\phi_\alpha^i,\ t^*F_\alpha^k+X_\alpha^k>_{R^k}$$

$$+<\phi^k_\alpha,\ t^*G^k_\alpha+Y^k_\alpha>_{R^k}+<w,\ t^*F^k_3+Z^k>_{R^k}\}$$

$$-2<\bar{u}^1_\alpha,\ t^*T^0_\alpha>_{R^1}-2<w,\ t^*T^0_3>_{R^1}+2<\bar{u}^1_\alpha,\ t^*T^n_\alpha>_{R^n}$$

$$+2<\sum_{i=1}^{n}t_i\phi^i_\alpha,\ t^*T^n_\alpha>_{R^n}+2<w,\ t^*T^n_3>_{R^n}$$

$$-2\sum_{k=1}^{n}\{\ <\bar{u}^1_\alpha,\ t^*\hat{N}^k_\alpha>_{S^k_1}+<\sum_{i=1}^{k-1}t_i\phi^i_\alpha,\ t^*\hat{N}^k_\alpha>_{S^k_1}+<\phi^k_\alpha,\ t^*\hat{M}^k_\alpha>_{S^k_3}$$

$$+<w,\ t^*\hat{Q}^k_\alpha\eta_\alpha>_{S^k_5}\} \tag{35}$$

which is the potential energy type variational principle for Sun's theory[9]. Equivalently, we can eliminate ϕ^k_α instead of $\bar{u}^k_\alpha$ and, then, it becomes the variational formulations of Srinivas' [10] or Seide's theory[11].

$$\Omega_3=\sum_{k=1}^{n}\{-<\bar{u}^k_\alpha,\ P^k\bar{u}^k_\alpha>_{R^k}-\frac{1}{t_k^2}<(\bar{u}^{k+1}_\alpha-\bar{u}^k_\alpha),I^k(\bar{u}^{k+1}_\alpha-\bar{u}^k_\alpha)>_{R^k}-<w,\ P^kw>_{R^k}$$

$$-<\bar{e}^k_{\alpha\beta},\ t^*A^k_{\alpha\beta\gamma\delta}\bar{e}^k_{\gamma\delta}>_{R^k}-\frac{1}{t_k^2}<(\bar{e}^{k+1}_{\alpha\beta}-\bar{e}^k_{\alpha\beta},\ t^*D^k_{\alpha\beta\gamma\delta}(\bar{e}^{k+1}_{\gamma\delta}-\bar{e}^k_{\alpha\beta})>_{R^k}$$

$$-\frac{2}{t_k}<\bar{e}^k_{\alpha\beta},\ t^*B^k_{\alpha\beta\gamma\delta}(\bar{e}^{k-1}_{\gamma\delta}-\bar{e}^k_{\gamma\delta})>_{R^k}-\frac{2}{t_k}<\bar{u}^k_\alpha,\ R^k(\bar{u}^{k+1}_\alpha-\bar{u}^k_\alpha>_{R^k}\}$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{n}<2e^k_{\alpha3},\ -2t^*\lambda^{kj}_{\alpha\beta}e^j_{\beta3}>_{R^k}$$

$$+2\sum_{k=1}^{n}\{\ <\bar{u}^k_\alpha,\ t^*F^k_\alpha+X^k_\alpha>_{R^k}+\frac{1}{t_k}<(\bar{u}^{k-1}_\alpha,\ t^*G^k_\alpha+Y^k_\alpha>_{R^k}$$

$$+<w,\ t^*F^k_3+Z^k>_{R^k}\}-2<\bar{u}^1_\alpha,\ t^*T^0_\alpha>_{R^1}-2<w,\ t^*T^0_3>_{R^1}$$

$$+2<\bar{u}^n_\alpha,\ t^*T^n_\alpha>_{R^n}+\frac{2}{t_k}<\bar{u}^{k+1}_\alpha-\bar{u}^k_\alpha,\ t^*t_nT^k_\alpha>_{R^k}+2<w,\ t^*T^n_3>_{R^n}$$

$$+\sum_{k=1}^{n}\{-2<\bar{u}^k_\alpha,\ t^*\hat{N}^k_\alpha)>_{S^k_1}-\frac{2}{t_k}<(\bar{u}^{k+1}_\alpha-\bar{u}^k_\alpha),\ t^*\hat{M}^k_\alpha)>_{S^k_3}-2<w,\ t^*\hat{Q}^k_\alpha\eta_\alpha)>_{S^k_5} \tag{36}$$

In connection with finite element formulation, it is worth noting that use of Ω_3 may be more convenient if in-plane stretching of individual layer needs to be specified. Clearly, a large number of specializations from other extended variational principles are possible even if they are not listed here, and some representative cases presented in Fig. 3.

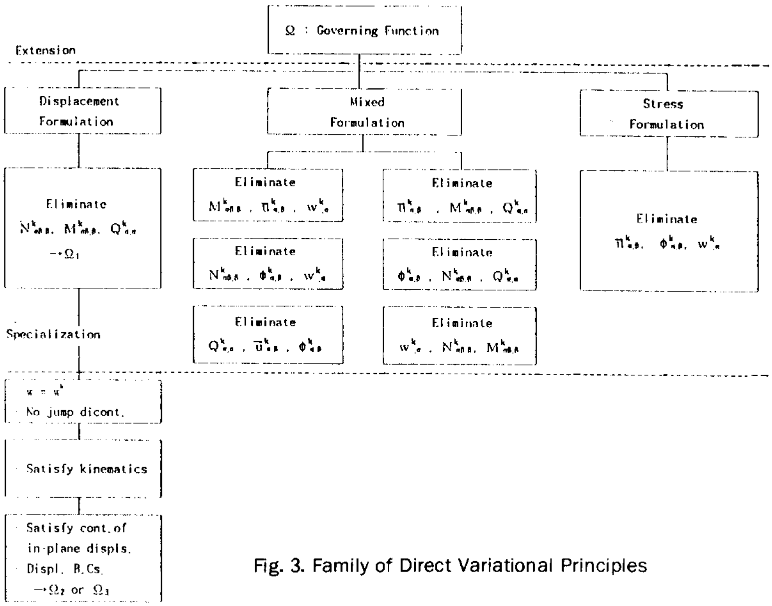

Fig. 3. Family of Direct Variational Principles

4. Complementary Formulations

An alternative procedure to set up variational principles governing the problem is to write the operator equations in complementary form instead of the direct formulation (26). Assuming the kinematic relations (18) are satisfied, the field equations (20)−(22) may be written in the self-adjoint form with respect to the bilinear mapping (27) as :

$$
\begin{bmatrix}
A_1 & B_1 & E_{1,2} & 0 & E_{1,3} & 0 & \cdots & E_{1,n-1} & 0 & E_{1,n} \\
 & 0 & C^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & & A_2 & B_2 & E_{2,3} & 0 & \cdots & E_{2,n-1} & 0 & E_{2,n} \\
 & & & 0 & C^T & 0 & \cdots & 0 & 0 & 0 \\
 & & & & A_3 & B_3 & \cdots & E_{3,n-1} & 0 & E_{3,n} \\
 & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & A_{n-1} & B_{n-1} & E_{n-1,n} \\
 & & & & & & & & 0 & C^T \\
 & & & & & & & & & A_n
\end{bmatrix}
\begin{Bmatrix}
U_1 \\ \Xi_1 \\ U_2 \\ \Xi_2 \\ U_3 \\ \Xi_3 \\ \cdot \\ \cdot \\ U_{n-1} \\ \Xi_{n-1} \\ U_n
\end{Bmatrix}
=
\begin{Bmatrix}
-r_1 + \Xi_0 \\ 0 \\ -r_2 \\ 0 \\ -r_3 \\ 0 \\ \cdot \\ \cdot \\ -r_{n-1} \\ 0 \\ -r_n - \Xi_n
\end{Bmatrix}
\tag{37}
$$

where

$$A_k = \begin{bmatrix} -P^k\delta_{\alpha\gamma} & -R^k\delta_{\alpha\gamma} & 0 & \frac{1}{2}t^*\Pi_1 & 0 & 0 \\ -R^k\delta_{\alpha\gamma} & -I^k\delta_{\alpha\gamma} & 0 & 0 & \frac{1}{2}t^*\Pi_1 & -t^* \\ 0 & 0 & -P^k & 0 & 0 & t^*\delta_{\alpha\gamma}\frac{\partial}{\partial\alpha} \\ -\frac{1}{2}t^*\Pi_2 & 0 & 0 & t^*\overline{A}^k_{\alpha\beta\gamma\delta} & t^*\overline{B}_{\alpha\beta\gamma\delta} & 0 \\ 0 & -\frac{1}{2}t^*\Pi_2 & 0 & t^*\overline{B}_{\alpha\beta\gamma\delta} & t^*\overline{D}_{\alpha\beta\gamma\delta} & 0 \\ 0 & -t^* & -t^*\delta_{\alpha\gamma}\frac{\partial}{\partial\alpha} & 0 & 0 & t^*\mu^{kk}_{\gamma\alpha} \end{bmatrix}$$

$$B_k^T = \begin{bmatrix} t^* & t^*t_k & 0 & 0 & 0 & 0 \\ 0 & 0 & t^* & 0 & 0 & 0 \end{bmatrix}$$

$$C_k^T = \begin{bmatrix} -t^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t^* & 0 & 0 & 0 \end{bmatrix}$$

$$E_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{ij}_{\gamma\alpha} \end{bmatrix} \qquad i,j = 1, 2, \ldots, n$$

$$U_k^T = [\overline{u}^k_\gamma, \phi^k_\gamma, w^k, N^k_{\gamma\delta}, M^k_{\gamma\delta}, Q^k_\alpha], \qquad k = 1, 2, \ldots, n$$

$$\Xi_k^T = [T^k_\alpha, T^k_3] \qquad k = 1, 2, \ldots, n-1$$

$$r_k^T = [t^*F^k_\alpha + X^k_\alpha, t^*G_\alpha + Y^k_\alpha, t^*F^k_3 + Z^k, 0, 0, 0]$$

$$\Xi_0^T = [t^*T^0_\alpha, 0, t^*T^0_3, 0, 0, 0]$$

$$\Xi_n^T = [t^*T^n_\alpha, t^*t_nT^n_\alpha, t^*T^n_3, 0, 0, 0]$$

$$\Pi_1 = \delta_{\alpha\delta}\frac{\partial}{\partial\gamma} + \delta_{\alpha\gamma}\frac{\partial}{\partial\delta}$$

$$\Pi_2 = \delta_{\alpha\gamma}\frac{\partial}{\partial\beta} + \delta_{\beta\gamma}\frac{\partial}{\partial\alpha}$$

Consistent boundary conditions associated with the operator equation (37) are :

$$
\begin{aligned}
&-t^{*}N^{k}_{\alpha\beta}\eta_{\beta} = -t^{*}\hat{N}_{\alpha} \quad \text{on } S^{k}_{1} \qquad -t^{*}M^{k}_{\alpha\beta}\eta_{\beta} = -t^{*}\hat{M}_{\alpha} \quad \text{on } S^{k}_{3}\\
&-t^{*}Q^{k}_{\alpha}\eta_{\alpha} = -t^{*}\hat{Q}^{k}_{\alpha}\eta_{\alpha} \quad \text{on } S^{k}_{5} \qquad t^{*}\bar{u}^{k}_{\alpha}\eta_{\beta} = t^{*}\hat{\bar{u}}^{k}_{\beta}\eta_{\alpha} \quad \text{on } S^{k}_{2}\\
&t^{*}\phi^{k}_{\alpha}\eta_{\beta} = t^{*}\hat{\phi}^{k}_{\alpha}\eta_{\beta} \quad \text{on } S^{k}_{4} \qquad t^{*}w^{k}\eta_{\alpha} = t^{*}\hat{w}^{k}\eta_{\alpha} \quad \text{on } S^{k}_{6}
\end{aligned} \tag{38}
$$

and the internal jump discontinuity conditions are :

$$
\begin{aligned}
&-t^{*}(N^{k}_{\alpha\beta}\eta_{\beta})' = -t^{*}(g^{k}_{1})_{\alpha} \quad \text{on } S^{k}_{1i} \qquad -t^{*}(M^{k}_{\alpha\beta}\eta_{\beta})' = -t^{*}(g^{k}_{3})_{\alpha} \quad \text{on } S^{k}_{3i}\\
&-t^{*}(Q^{k}_{\alpha}\eta_{\alpha})' = -t^{*}(g^{k}_{5i})_{\alpha}\eta_{\alpha} \quad \text{on } S^{k}_{5i} \qquad t^{*}(\bar{u}^{k}_{\alpha}\eta_{\beta})' = t^{*}(g^{k}_{2})_{\alpha}\eta_{\beta} \quad \text{on } S^{k}_{2i}\\
&t^{*}(\phi^{k}_{\alpha}\eta_{\beta})' = t^{*}(g^{k}_{4})_{\alpha}\eta_{\beta} \quad \text{on } S^{k}_{4i} \qquad t^{*}(w^{k}\eta_{\alpha})' = t^{*}(g^{k}_{6})\eta_{\alpha} \quad \text{on } S^{k}_{6i}
\end{aligned} \tag{39}
$$

By the definitions (4), the following relations between off-diagonsl operatord are satisfied.

$$
\begin{aligned}
<\bar{u}^{k}_{\alpha}, t^{*}N^{k}_{\alpha\beta,\beta}>_{R^{k}} &= -<t^{*}\bar{u}^{k}_{\alpha,\beta}, N^{k}_{\alpha\beta}>_{R^{k}} + <\bar{u}^{k}_{\alpha}, t^{*}N^{k}_{\alpha\beta}\eta_{\beta}>_{S^{k}_{1}} + <N^{k}_{\alpha\beta}, t^{*}\bar{u}^{k}_{\alpha}\eta_{\beta}>_{S^{k}_{2}}\\
&\quad + <\bar{u}^{k}_{\alpha}, t^{*}(N^{k}_{\alpha\beta}\eta_{\beta})'>_{S^{k}_{1i}} + <N^{k}_{\alpha\beta}; t^{*}(\bar{u}^{k}_{\alpha}\eta_{\beta})'>_{S^{k}_{2i}}\\
<\phi^{k}_{\alpha}, t^{*}M^{k}_{\alpha\beta,\beta}>_{R^{k}} &= -<t^{*}\phi^{k}_{\alpha,\beta}, M^{k}_{\alpha\beta}>_{R^{k}} + <\phi^{k}_{\alpha}, t^{*}M^{k}_{\alpha\beta}\eta_{\beta}>_{S^{k}_{3}} + <M^{k}_{\alpha\beta}, t^{*}\phi^{k}_{\alpha}\eta_{\beta}>_{S^{k}_{4}}\\
&\quad + <\phi^{k}_{\alpha}, t^{*}(M^{k}_{\alpha\beta}\eta_{\beta})'>_{S^{k}_{3i}} + <M^{k}_{\alpha\beta}, t^{*}(\phi^{k}_{\alpha}\eta_{\beta})'>_{S^{k}_{4i}}\\
<w^{k}, t^{*}Q^{k}_{\alpha,\alpha}>_{R^{k}} &= -<t^{*}w^{k}_{,\alpha}, Q^{k}_{\alpha}>_{R^{k}} + <w^{k}, t^{*}Q^{k}_{\alpha}\eta_{\alpha}>_{S^{k}_{5}} + <Q^{k}_{\alpha}, t^{*}w^{k}\eta_{\alpha}>_{S^{k}_{6}}\\
&\quad + <w^{k}, t^{*}(Q^{k}_{\alpha}\eta_{\alpha})'>_{S^{k}_{5i}} + <Q^{k}_{\alpha}, t^{*}(w^{k}\eta_{\alpha})'>_{S^{k}_{6i}}
\end{aligned} \tag{40}
$$

For the matrix equation (37), the governing functional is obtained, following (6), as

$$
\begin{aligned}
J = &\sum_{k=1}^{n} <U^{T}_{k}, A_{k}U_{k}>_{R^{k}} + \sum_{k=1}^{n-1} <U^{T}_{k}, B_{k}\Xi_{k}>_{R^{k}} + \sum_{k=2}^{n} <U^{T}_{k}, C\,\Xi_{k-1}>_{R^{k}}\\
&+ \sum_{k=1}^{n}\sum_{j=1}^{n} <U^{T}_{k}, E_{j,k}U_{j}>_{R^{k}} - \sum_{k=1}^{n} <U^{T}_{k}, E_{k,k}U_{k}>_{R^{k}}
\end{aligned}
$$

$$+\sum_{k=1}^{n-1} \langle \Xi_k^T, B_k^T U_k \rangle_{R^k} + \sum_{k=1}^{n-1} \langle \Xi_k^T, C^T U_{k+1} \rangle_{R^k}$$

$$+2\sum_{k=1}^{n} \langle U_k^T, r_k \rangle_{R^k} - 2\langle U_1^T, \Xi_0 \rangle_{R^1} + 2\langle U_n^T, \Xi_n \rangle_{R^n}$$

$$+\text{Boundary Terms} + \text{Internal Jump Terms} \qquad (41)$$

Using Seide's discrete laminate theory[11], Al-Gothani[12] presented the complementary formulation of laminated composite plate for the dynamic case. The formulation is the same as the one given above, except for that the operator matix in (36) includes matrix $E_{i,j}$ representing the coupling of transverse shear constitutive relations between layers based upon the consistent shear deformable theory[13]. Consequently, if this terms are taken account of, most discussions given by him is applicable to the present formulation. Thus, extensions and specializations of the general variational principle (40) are not repeated here, but some ways of its extensions and specializations which lead to interesting form of variational principles are presented in Fig. 4.

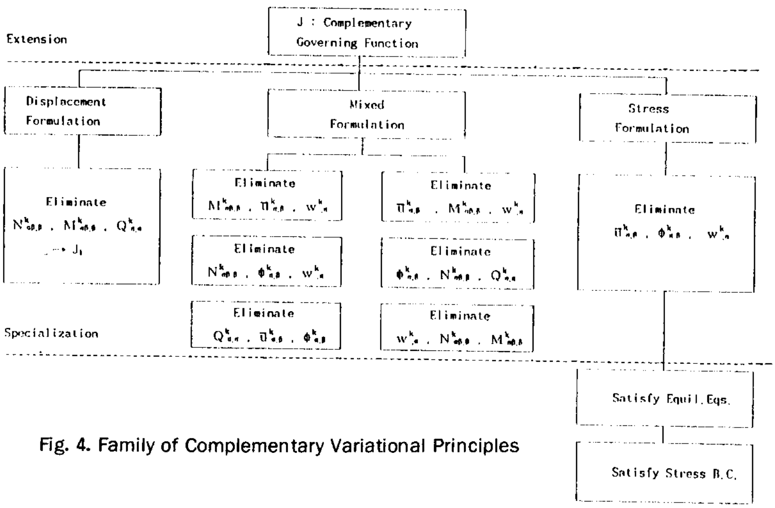

Fig. 4. Family of Complementary Variational Principles

5. Discussion

Based on the discrete laminated plate theory, which accounts for the effect of transverse shear deformation in a consistent manner, a systematic development of variational principles for dynamics of linear elastic composite laminated plate has been presented. The direct as well as complementary formulations are considered. Complementary self-adjoint form of the field equations is the same as the one presented by Al-Gothani[12], except for the coupling terms of transverse shear constitutive equations between layers which have been introduced in the consistent shear deformable theory[13]. Nonhomogeneous boundary conditions and internal jump discontinuity have been explicitly included in general variational principle. Allowance of jump discontinuity terms in variational formulation is meaningful in the context of direct approximation in finite element spaces since the space of approximants may not be sufficiently smooth. Also, extensions of the general variational principle and specializations by restricting some of the field equations and/or boundary conditions to be identically satisfied have been discussed and depicted diagrammatically in Fig. 3 and 4, respectively, for the direct and complementary formulation. These formulations should provide a basis for the development of alternative approaches to approximate solution of the problem and also for the development of approximate theories.

References

1. Mikhlin, S. G., *The Problem of the Minimun of a Quadratic Functional*, Holden-Day, San Francisco, 1965.
2. Gurtin, S. G., "Variational Principles in the Linear Theory of Viscoelasticity", *Arch. Rat. Mech. Anal.*, Vol=13, 179-191, 1963.
3. Gurtin, S. G., "Variational Principles in the Linear Elastodynamics", *Arch. Rat. Mech. Anal.*, Vol=16, 234-254, 1964.
4. Prager, W., "Variational principles for Elastic Plates with Relaxed Continuity Requirements", *Int. J. Solids Struct.*, vol.4, 837-844, 1968.
5. Sandhu, R. S. and Pister, K. S., "A Variational Principle for Linear Coupled Field Problems", *Int. J. Eng. Sci.*, Vol=8, 989-999, 1970.
6. Sandhu, R. S. and Pister, K. S., "Variational Principles for Boundary Value and Initial Boundary Value Problems", *Int. J. Solids Struct.*, Vol=7, 639-654, 1971.
7. Sandhu, R. S. and Salaam, U., "Variational Formulation of Linear Problems with Nonhomogeneous Boundary Conditions and Internal Discontinuities", *Comp. Meth. Appl. Mech. Eng.*, Vol=7, 75-91, 1975.
8. Sandhu, R. S., "Variational Principles for Finite Element Approximations", in *Finite Element in Water Resources Engineering*, G. F. Pinder and C. A. Brebbia(Eds.), Pentech Press, 1976.
9. Sun, C. T. and Whitney, J. M., "Theories for the Dynamic Response of Laminated Plates", *AIAA J.*, Vol=11, 178-183, 1973.
10. Srinivas, S., "Refined Analysis of Composite Laminates", *J. Sound and Vib.*, Vol=30, 495-507, 1973.
11. Seide, P., "Approximate Theory for the Bending of Laminated Plates", in *Mechanics Today*, N. Nasser(ed.), Vol=5, 451-466, 1980.
12. Al-Ghothani, A. M., *A Unified Approach to the Dynamics of Bending and Extension of Moderately Thick Laminated Composite Plate*, Ph. D Dissertation, The Ohio State Univ., Columbus, Ohio, 1986.
13. Hong, S. J. and Sandhu, R. S., "Consistent Shear Constitutive Relations for Laminated Plates", *ASTM J.* Composite Technology and Research, Vol=23, 205-220, 1992.

(접수일자 1994. 7. 13)