안전 구조물의 퍼포먼스 측정시 나타나는 Euler 방정식의 수치해석적 안정성

Numerical Experiments on the Stability of Euler Equations of the Performance Test of Safety Structures

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요 약

도로 안전 구조물을 설계하고 동적 퍼포먼스를 측정하기 위하여 시행되는 충돌시험시 충돌차량의 운동량, 특히 각가속도를 Euler 방정식의 수치적분을 통하여 구하는 기법을 설명하였다.

수치적으로 가장 안정적인 9-array 시스템안에 내재된 여러 형태의 7, 8-array 서브시스템들의 시스템 미분 방정식 및 이들 방정식의 수치 적분시 안정성을 실험데이터를 이용하여 파악하였다.

기본적인 9-array 안에 있는 6개의 8-array 시스템들은 모두 수치적으로 안정성을 보였고 12개의 7-array 시스템들중 6개는 안정성을 보이고 6개는 불안정하였다. 안정성을 보이는 내재된 서브시스템을 활용하면 기본 9-array 시스템을 구성하는 센서의 일부 고장시 효과적으로 각가속도를 측정할 수 있는 fail-safe 시스템을 구성할 수 있다.

Abstract

To design and study the dynamic performance of safety structures, crash tests are needed. Method to get the angular accelerations at the time of impact by integating the Euler equations are introduced. Numerically stable 9-array system contains several 7 and 8-array sub-systems in it. Numerical stability of those latent sub-systems are studied using test files. All of the 8-array sub-systems were found to be numerically stable. Six of the 7-array sub-systems were stable and other six of the 7-array sub-systems were unstable. Using this findings fail-safe measurement system can be developed.

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Introduction

To design and assess the dynamic performance of the roadside safety structures, consideration must be given to the dynamic response of vehicle and their occupants during the interaction with roadside strutures such as guardrails, bridge rails, median barriers, crash cushions, sign posts and utility poles etc. The National Corporrative Highway Research Program(NCHRP) report 230(Michie 1981) is the document currently used in the USA to evaluate the performance of safety features. The concept of evaluation in the report is that the dynamics of the vehicle are measured and injury descriptors such as relative impact velocity with which the occupants impacts the vehicle interior and maximum deceleration experienced by the vehicle after the occupant impact are calculated. These injury descriptors are calculated by the flail space model which is two dimensional in nature and has the limitation to describe the detailed interaction of occupant and impacting vehicle.

For the better understanding of the mechanism of interaction between the occupant and impacting vehicle, it is important to measure the vehicular motion precisely at its impact against the safety structures. If we consider the impact vehicle as a rigid body, six degree of freedom i.e., three translational acclerations and three angular accelerations are needed to describe the motion completely.

Transducer to mearsure the translational accelertion is available. But the angular acceleration is not electronically measurable. Many researchs have been conducted to solve the problem and the reliable and-widely used method to find angular acceleration is to strategically place the translational transducers(accelerometers) on the rigid body and use the Euler equations. This technique is called the accelerometry. It is widely known that minimum 6 accelerometers are needed to set up the system differential equation for the angular accelerations. This method, however, is known to be numerically unstable for a certain motion because of the complexity of the resulting equation. On the other hand, if 9 accelerometers are used, resulting system differential equation become simpler and the numerical instability will not be a problem (Padagaonka 1975).

One point to be noted in the 9 accelerometer system is that the system comprises several number of 7 and 8 accelerometer system. In this study, the method to find the system differential equations of the 6 and 9 accelerometer systems will be presented. Also the system equation of the 7 and 8 accelerometer system latent in the basic 9 array will be found and the numerical stability of the resulting system differential equations will be studied using some of the Texas Transportation Institute(TTI) test files. In the study, possible measurement errors will be disregarded and TTI test files will be regarded as error-free kinematic quantities.

Analytical Formulations

Considering there is no movement of a point P relative to body fixed coordinates, the absolute acceleration of a point P on a rigid body (Fig. 1) will be obtained by the following Euler equations(Beer 1988)

$$\ddot{r}=\ddot{R}+\ddot{\rho}$$
$$=\ddot{R}+\dot{\omega}\times\rho+\omega\times(\omega\times\rho) \tag{1}$$

This vector equation has three component equations as follows :

$$A_{ix}=A_{ox}+\omega_x\omega_y\rho_{iy}+\omega_x\omega_z\rho_{iz}-(\omega_y^2+\omega_z^2)\rho_{ix}+\dot{\omega}_y\rho_{iz}-\dot{\omega}_z\rho_{iy} \quad (2.a)$$

$$A_{iy}=A_{oy}+\omega_x\omega_y\rho_{ix}+\omega_y\omega_z\rho_{iz}-(\omega_x^2+\omega_z^2)\rho_{iy}-\dot{\omega}_x\rho_{iz}+\dot{\omega}_z\rho_{ix} \quad (2.b)$$

$$A_{iz}=A_{oz}+\omega_x\omega_z\rho_{ix}+\omega_y\omega_z\rho_{iy}-(\omega_x^2+\omega_y^2)\rho_{iz}+\dot{\omega}_x\rho_{iy}-\dot{\omega}_y\rho_{ix} \quad (2.c)$$

where

· A_{ix}, A_{iy}, A_{iz} : x, y, z components of translational acceleration at point i

· A_{ox}, A_{oy}, A_{oz} : components of translational acceleration at the origin of rigid body fixed coordinate system

· $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$: angular accelerations

· ω_x, ω_y, ω_z : angular velocity components

· ρ_{ix}, ρ_{iy}, ρ_{iz} : coordinates of a point i in the body fixed coordinate system.

It is possible to measure the angular accelerations using Eqs. 2.a, b, c by way of several techniques.

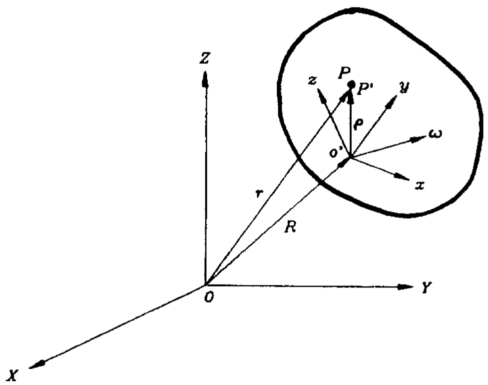

Fig. 1 Rigid Body Motion in a Moving Coordinate System

Six Accelerometer Array(3-2-1 Configuration)

Consider the six accelerometer configuration in Fig. 2. In this configuration three accelerometers are placed at the origin(0, 0, 0) in the x, y and z direction ; two at (ρ_{1x}, 0, 0) in the y and z direction and one at (0, ρ_{2y}, 0) in the z directions. These 6 linear transducers will give the readings of A_{ox}, A_{oy}, A_{oz}, A_{1y}, A_{1z} and A_{2z}. Substituting the location coordinates of each transducer into Eq. 2, A_{1z}, A_{1y} and A_{2z} can be written as follows :

$$\begin{aligned} A_{1z}&=A_{oz}+\omega_x\omega_z\rho_{1x}-\dot{\omega}_y\rho_{1x}\\ A_{1y}&=A_{oy}+\omega_x\omega_y\rho_{1x}+\dot{\omega}_z\rho_{1x}\\ A_{2z}&=A_{oz}+\omega_y\omega_z\rho_{2y}-\dot{\omega}_x\rho_{2y}\end{aligned} \quad (3)$$

These will yield the following three equations for angular accelerations :

$$\begin{aligned} \dot{\omega}_x&=\frac{(A_{2z}-A_{oz})}{\rho_{2y}}-\omega_y\omega_z\\ \dot{\omega}_y&=\frac{(A_{oz}-A_{1z})}{\rho_{1x}}+\omega_x\omega_z\\ \dot{\omega}_z&=\frac{(A_{1y}-A_{oy})}{\rho_{1x}}-\omega_x\omega_y\end{aligned} \quad (4)$$

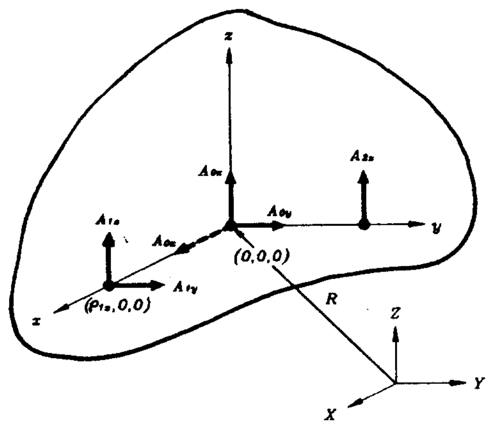

Fig. 2 Six Accelerometer Array

It can be seen that the angular accelerations in Eq. 4 require prior knowledge of the angular velocities about x, y and z axes. These nonlinear, coupled ordinary differential equation can be solved numerically using the Euler method, Runge-Kutta method, etc.. A_{ox}

is not used in the equation but it is necessary to construct an acceleration field of the rigid body. A_{ox}, A_{oy} and A_{oz} are used as the translational acceleration at a reference point. This is the idea which makes 3-2-1 array. For this system to work, the resulting differential equation must be numerically solvable for angular accelerations without any significant errors. Experimentally it has been shown that the system equations of the 3-2-1 array is not stably integrable for angular accelerations. This problem is due to the coupled nonlinear terms in the equations. To solve the problem, attempts have been made to removė the coupled terms by placing more translational accelerometers and 9 accelerometer system is a success(Padagaonka 1975).

Nine Accelerometer array(3-2-2-2 Configuration) :

An alternative way to circumvent the difficulties encountered during the numerical integration of the equation resulting from 6-array accelerometry is to use a nine accelerometer array as shown in Fig. 3. This new configuration is formed by adding three accelerometers A_{3x}, A_{3y} and A_{1x} to the 3-2-1 configuration. These three added accelerometers will give three additional nonlinear coupled ordinary differential equations. By algebraically manipulating these three equations with existing ones, coupled nonlinear angular velocity terms can be removed.

The three additional acceleration readings will give three additional differential equations similar to Eq. 3. Considering the location coordinates, Eq. 2 will give the following 3 equations :

$$
\begin{aligned}
A_{3y} &= A_{oy} + \omega_y \omega_z \rho_{3z} - \dot{\omega}_x \rho_{3z} \\
A_{3x} &= A_{ox} + \omega_x \omega_z \rho_{3z} + \dot{\omega}_y \rho_{3z} \\
A_{2x} &= A_{ox} + \omega_x \omega_y \rho_{2y} - \dot{\omega}_z \rho_{2y}
\end{aligned}
\qquad (5)
$$

Then Eq. 5 can be rewritten in a form similar to Eq. 4 as follows :

$$
\begin{aligned}
\dot{\omega}_x &= \frac{(A_{oy} - A_{3y})}{\rho_{3z}} + \omega_y \omega_z \\
\dot{\omega}_y &= \frac{(A_{3x} - A_{ox})}{\rho_{3z}} - \omega_x \omega_z \\
\dot{\omega}_z &= \frac{(A_{ox} - A_{2x})}{\rho_{2y}} + \omega_x \omega_y
\end{aligned}
\qquad (6)
$$

By algebraically adding Eq. 4 to Eq. 6, we can eliminate the cross products of the angular velocity components from the differential Eqs. 4 and 6. Then the angular acceleration can be calculated without reliance on values of parameters calculated at the previous time step. This is the idea of the 9 accelerometer array(3-2-2-2 configuration). The resulting equations for angular accelerations are as follows :

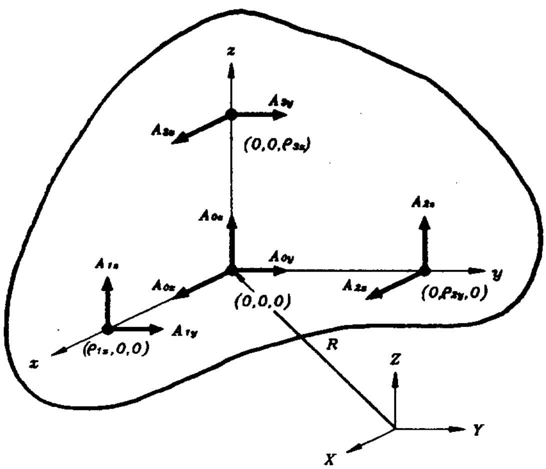

Fig. 3 Nine Accelerometer Array

$$\dot{\omega}_x = \frac{(A_{2z} - A_{0z})}{2\rho_{2y}} + \frac{(A_{0y} - A_{3y})}{2\rho_{3z}}$$
$$\dot{\omega}_y = \frac{(A_{0z} - A_{1z})}{2\rho_{1x}} + \frac{(A_{3x} - A_{0x})}{2\rho_{3z}} \qquad (7)$$
$$\dot{\omega}_z = \frac{(A_{1y} - A_{0y})}{2\rho_{1x}} + \frac{(A_{0x} - A_{2x})}{2\rho_{2y}}$$

Latent Systems & Numerical Experiments

It should be noted that the 3-2-2-2 array comprises several different forms of seven and eight array systems. The possible latent systems and corresponding system equations are in Appendix A.

System equation are derived using Eq. 2 and following the procedures explained in the previous section.

The naming of the latent system is composed of one number followed by X, Y or Z. The first number represents the node on which the seismic center of a transducer is located. For example, '0' means the transducer is located at the origin of the body fixed coordinate system, '1' means the node is on the x axis, '2' on the y-axis, '3' on the z-axis respectively. Following X, Y, Z represents the principle sensitivity axis of the transducer along which the accelerations are to be measured. In this study, the ID represents the missing accelerometer from the basic 9 accelerometer array(3-2-2-2 configuration). For example, 1Y represents an eight array system which is same as 9 array(3-2-2-2 configuration) except the fact that the transducer at node 1 in the Y direction dose not exist. Similarly, if the system ID is composed of "NUMBER AXIS NUMER AXIS", it means two accelerometers are missing from the basic 9-array system. For example, 1Y1Z is a seven array system which is same as 9 array system (3-2-2-2 configuration) except the fact that transducers at node 1 in the Y and Z directions do not exist.

The main purpose of this study is to check whether the resulting system equations of the latent system are stably integrable or not.

For the study, 6 number of TTI test files, 7110-10, 7110-12, 7110-4, 7110-2(Ross et al, 1991), 7043-2, 7043-1(Ross et al, 1988) were used. Those test files have significant meaning in studying the numerical stability of the system differential equations. Beacuse, if we use a hypothetically well behaved functions like sinusoidal functions in the stabilty study, it may give a stable output which is not the case for the experimental data. To study the stability of the system equations, therefore, it is important to use a signal which is similar to the output of the crash tests performed for the design of safety structures. TTI measured 3 translational accelerations and three angular velocities but not angular accelerations. In this study, angular velocities are differentiated and regarded as measured angular accelerations. This process is not recommendable since the process of differentiation a "noisy" data is basically an unstable process-meaning that small errors made during the process cause greatly magnified errors in the final results. But in the numerical experiment, assume that the differentiation of measured angular velocity is the way to get the angular acceleration which represents the crash test enviromentals most closely.

The procedure to check the numerical stability is as followes :

STEP 1. For each of the system equation of the latent systems, move all the angular velocity terms to the left hand side

SETP 2. Caculate the angular acceleration or angular acceleration plus angular velocity coupled terms using the TTI test files. Those values will be regarded as the algebraic sum of the transducer readings which appears on the right side of the system equations.

STEP 3. Now, all the unknowns are on the left hand side and the right hand side is the known constant. These are typical nonlinear first order coupled differential equations. In solving the problems for angular velocity and angular accelerations, Euler method or Runge-Kutta method can be used. In this study R-K 4th order method will be used.

STEP 4. Compare the calculated angular accelerations with the angular accelerations of the TTI test files. If they overlay each other, the system can be regarded as numerically stable, otherwise not.

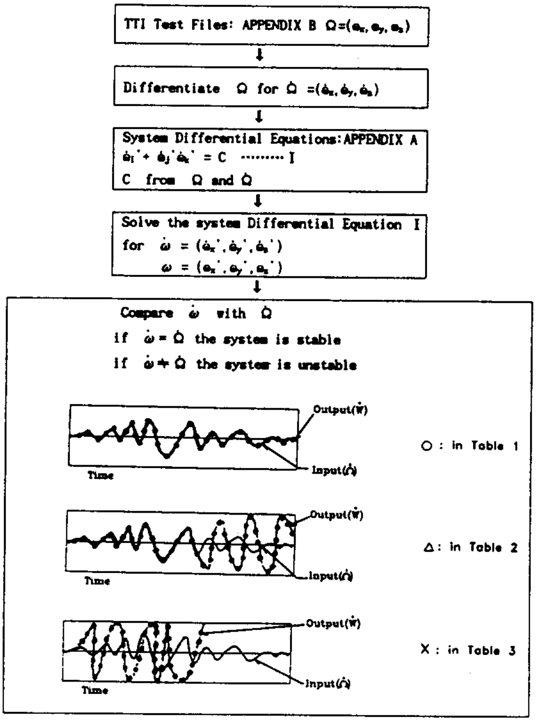

Fig. 4 Flow Chart of System Fidelity Study

TTI test files used for the numerical experiments are in Appendix B. They cover wide range of time duration, amplitude and frequency of various kinds of crash test environments. The results of the experiments are summerized in Table 1. In the table, 'o' represents a stable system, 'x' represents unstable ones and 'Δ' represents stable results with some instabilities after a certain period of time. From the summary table, it can be noted that some of the seven accelerometer systems are stable and some are not, and all the eight accelerometer system are stable. The systems that showed consistent stabilities among 7 array are 1Y1Z, 2X2Z, 3X3Y, 1Y3Y, 1Z2Z and 2X3X. Other system showed instabilities consistantly.

Table 1. Summary of Fidelity Study of 7, 8 and 9 array System(Runge-Kutta 4th order, Time step : 0.001sec)

I.D.	7110-10	7110-12	7110-4	7110-2	7043-1	7043-2
1Y1Z	○	○	○	○	○	○
2X2Z	○	○	○	○	○	○
3X3Y	○	○	○	○	○	○
1Y2Z	×	×	×	×	×	×
1Y3X	×	×	×	×	△	△
1Y3Y	○	○	○	○	○	○
1Z2X	×	×	×	×	△	△
1Z2Z	○	○	○	○	○	○
1Z3Y	×	×	×	×	×	×
2X3X	○	○	○	○	○	○
2X3Y	×	×	×	×	×	×
2Z3X	×	×	×	×	×	×
2X	○	○	○	○	○	○
2Z	○	○	○	○	○	○
3Y	○	○	○	○	○	○
3X	○	○	○	○	○	○
1Y	○	○	○	○	○	○
1Z	○	○	○	○	○	○
9-ARRAY (3-2-2-2)	○	○	○	○	○	○

Conclusion & Recommendations

To design and study the dynamic performance of safety structures, crash tests are needed.

Method to get the angular accelerations at the time of impact by integating the Euler equations are introduced. Minimum 6 transducers are necessary to set up the system differential equations. The 6-array system is known to be unstable due to the angular velocity coupled terms. By adding 3 trandsducers to the 6-array system, coupled terms can be removed and system becomes stable. This is the 9-array system(3-2-2-2 configuration).

In this study, 7 and 8-array systems latent in the basic 9-array systems are identified.

System differential equations of those latent sub-systems are introduced and numerical stability of the equations are studied using seven TTI test files.

All of the 8-array sub-systems(1Y, 1Z, 2X, 2Z, 3Y, 3X) were found to be numerically stable. Six of the 7-array sub-systems(1Y1Z, 2X2Z, 3X3Y, 1Y3Y, 1Z2Z, 2X3X) were stable and other six of the 7-array sub-systems(1Y2Z, 1Y3X, 1Z2X, 1Z3Y, 2X3Y, 2Z3X) were unstable. Using this findings fail-safe measurement system can be developed to measure the angular accelerations when one or two of the trandsducers in the basic 9-array system are in trouble or malfunction.

This finding is limited to the perpect system. In actual system, transducers may have certain kind of deterministic errors and system may be misaligned. If those errors are considered, system differential equations will be different from error-free case. Developing a mathmatical model to properly consider those errors and finding a strategy to remove those errors in the experiment will be a valuable future research.

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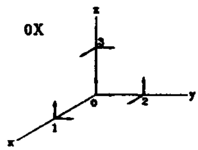
Appedix A. Seven and Eight Latent Systems in the Basic 9-array(3-2-2-2 Configuration) System

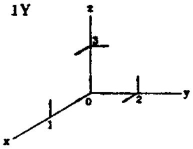
A. Eight Accelerometer Array Systems Latent in the 3-2-2-2 System

0X



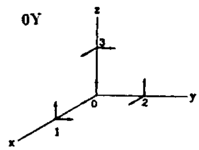
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

1Y



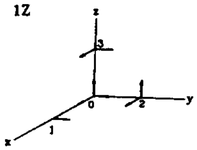
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 + \omega_x\omega_y$$

0Y



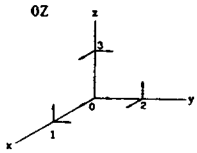
$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 + \omega_x\omega_y$$

1Z



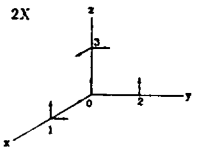
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

0Z



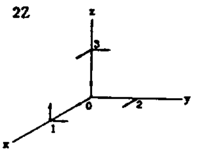
$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 + \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 - (A_{0x} - A_{2x})/2\rho_2$$

2X



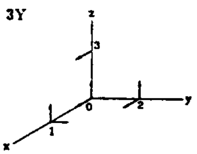
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

2Z



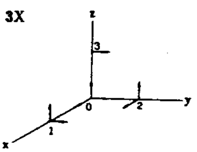
$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 + \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

3Y



$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

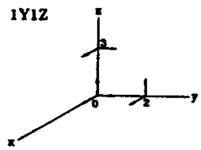
3X



$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

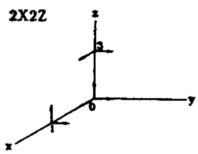
B. Seven Accelerometer Array Systems Latent in the 3-2-2-2 System

1Y1Z



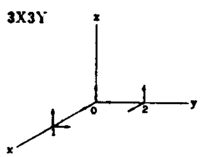
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 + \omega_x\omega_y$$

2X2Z



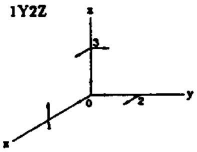
$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 + \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

3X3Y



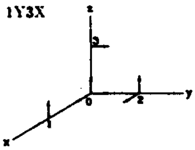
$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 - (A_{0x} - A_{2x})/2\rho_2$$

1Y2Z



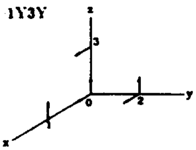
$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 + \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 - \omega_x\omega_y$$

1Y3X



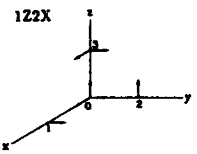
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 + \omega_x\omega_y$$

1Y3Y



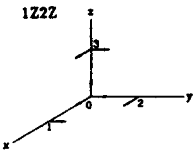
$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 - (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{0x} - A_{2x})/\rho_2 - \omega_x\omega_y$$

1Z2X



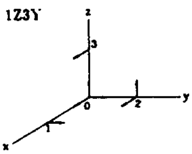
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

1Z2Z



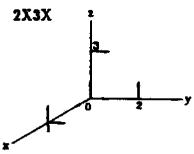
$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 + \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 - (A_{0x} - A_{2x})/2\rho_2$$

1Z3Y



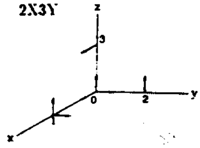
$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{3x} - A_{0x})/\rho_3 - \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

2X3X



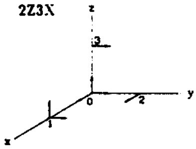
$$\dot{\omega}_x = (A_{2z} - A_{0z})/2\rho_2 + (A_{0y} - A_{3y})/2\rho_3$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

2X3Y



$$\dot{\omega}_x = (A_{2z} - A_{0z})/\rho_2 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/2\rho_1 + (A_{3x} - A_{0x})/2\rho_3$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/\rho_1 - \omega_x\omega_y$$

2Z3X



$$\dot{\omega}_x = (A_{0y} - A_{3y})/\rho_3 - \omega_y\omega_z$$
$$\dot{\omega}_y = (A_{0z} - A_{1z})/\rho_1 + \omega_x\omega_z$$
$$\dot{\omega}_z = (A_{1y} - A_{0y})/2\rho_1 + (A_{0x} - A_{2x})/2\rho_2$$

Appendix B. TTI Test Files Used for the Numerical Experiments

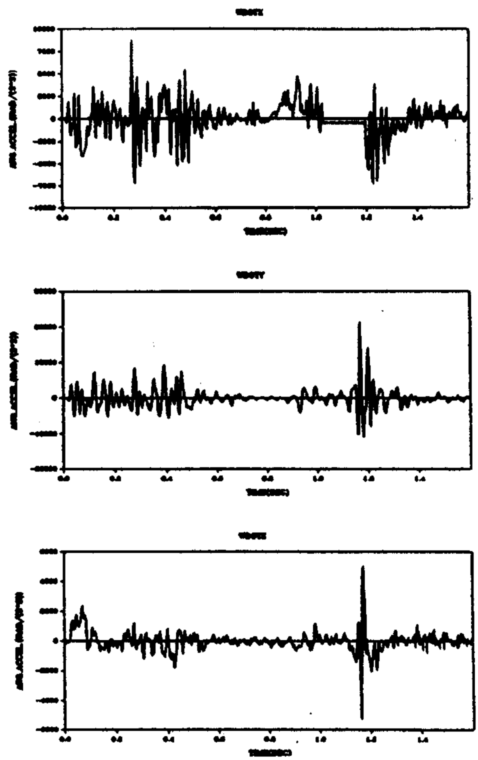


FIGURE B. 1 Input Data : 7110-10 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$

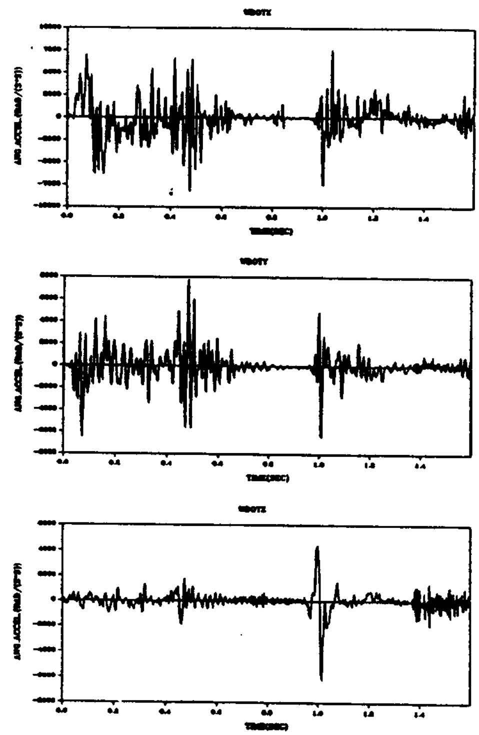


FIGURE B. 2 Input Data : 7110-12 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$

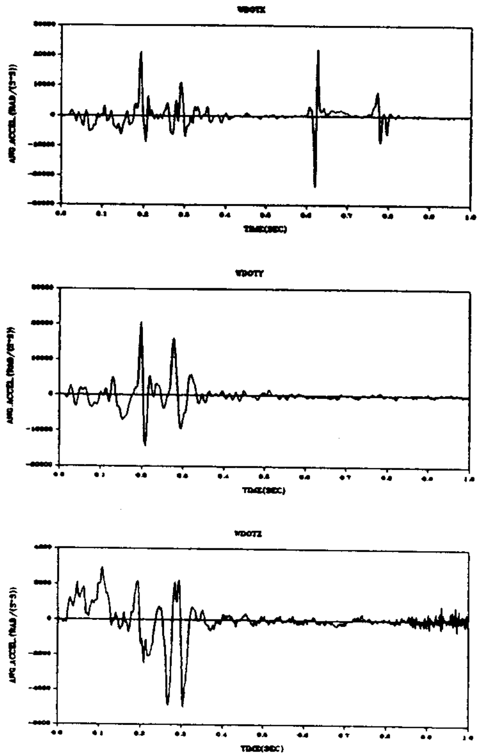


FIGURE B. 3 Input Data : 7110-4 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$

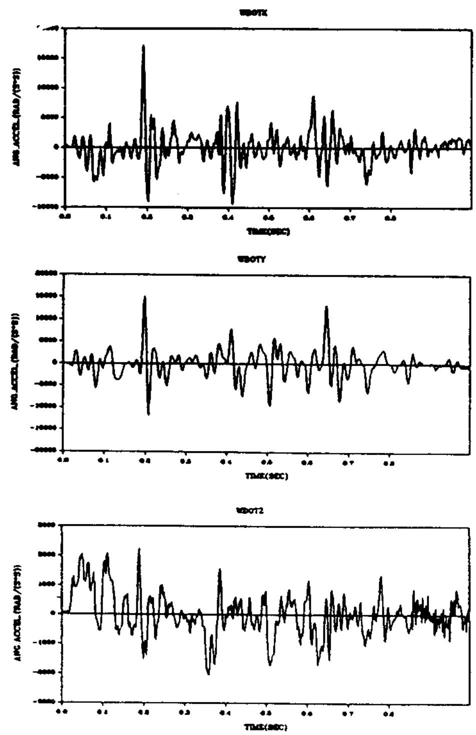


FIGURE B. 4 Input Data : 7110-2 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$

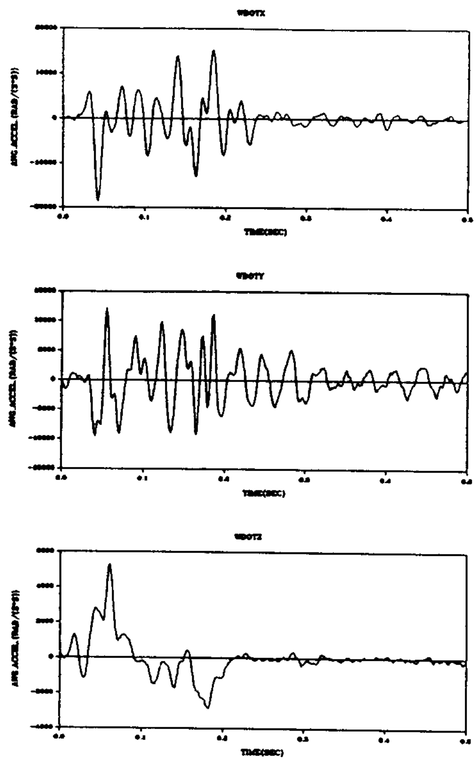


FIGURE B. 5 Input Data : 7043-1 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$

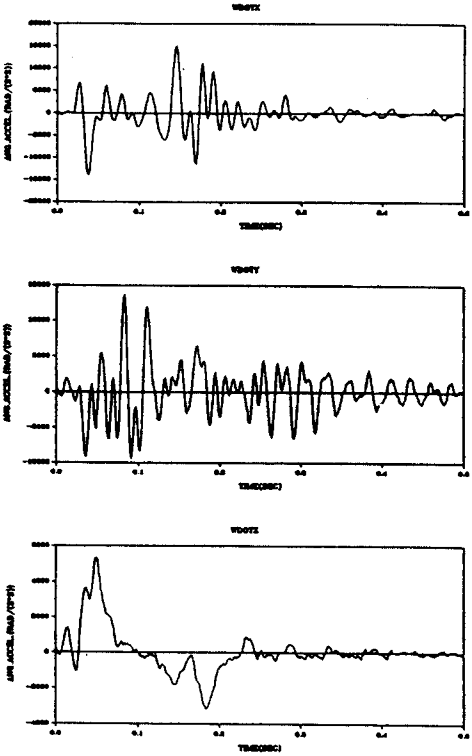


FIGURE B. 6 Input Data : 7043-2 $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$