Journal of the Korean
OR/ MS Society
Vol. 19, No. 3, December 1994

Waiting Times in the B/G/1 Queue with Server Vacations

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Abstract

We consider a B/G/1 queueing system with vacations, where the server closes the gate when it begins a vacation. In this system, customers arrive according to a Bernoulli process. The service time and the vacation time follow discrete distributions. We obtain the distribution of the number of customers at a random point in time, and in turn, the distribution of the residence time (queueing time + service time)for a customer. It is observed that solutions for our discrete time B/G/1 gated vacation model are analogous to those for the continuous time M/G/1 gated vacation model.

1. Introduction

We consider a discrete time B/G/1 gated queueing system with vacations where the server closes the gate when it *begins* a vacation. Analytical models exist for the continuous time M/G/1 gated queueing system with vacations where the gate closes when the server *ends* a vacation [3]. It is intuitively appealing that two different gate-closing policies(where the gate closes when the server *ends* a vacation. vs. when it *begins* a vacation) will make no difference in performance measures and that the continuous time model and the discrete time model will result in analogous solutions. However, to our knowledge, the system we consider here has not been analyzed before. This motivated our study.

Time is divided into consecutive fixed-length slots. Customer arrivals are assumed to occur at the beginning of each time slot according to a Bernoulli process. The service time and vacation time are limited to take only nonnegative integer multiples of a slot time. The system is assumed to have infinite buffer with FCFS queueing discipline. We obtain the distribution of the number of customers at a random point in time, and in turn, the distribution of the residence time(queueing time+service time) for a customer.

2. Analysis

The gated service mechanism we consider is different from that usually considered in the literature in which the server closes the gate when it returns from a vacation and serves the customers that arrived during the previous cycle. In our model, the server closes the gate ***when it begins a vacation.*** That is, if we number the sequence of vacations and service periods as (v_n, t_n), $n=1,\cdots$, then the customers arriving during vacation v_n and service period t_n are served during t_{n+1}.

It is well known that for several specific models of an M/G/1 queue with vacations, the following property holds [3].

M/G/1 Decomposition Property. The stationary number of customers present in the system at a random point in time is distributed as the sum of two or more independent random variables, one of which is the stationary number of customers present in the corresponding standard M/G/1 queue (i.e., the server is always available) at a random point in time.

Fuhrmann and Cooper demonstrate that this decomposition property also holds for a very general class of M/G/1 queueing models including the standard gated system [2]. It can easily be seen that the M/G/1 decomposition property also holds for our B/G/1 gated model. The result obtained by Fuhrmann and Cooper is stated in the following Lemma.

Lemma

$$g(z) = f(z)\frac{1-\alpha(z)}{\alpha'(1)(1-z)}\pi(z), \tag{1}$$

where $g(\cdot)$ = the p.g.f. for the stationary distribution of the number of customers that a random departing customer leaves behind in the vacatin system,

$f(\cdot)$ = the p.g.f. for the stationary distribution of the number of customers already present when a vacation begins,

$\alpha(\cdot)$ = the p.g.f. for the stationary distribution of the number of customer that arrive during a vacation,

$\pi(\cdot)$ = the p.g.f. for the stationary distribution of the number of customer that a random departing customer leaves behind in the corresponding standard M/G/1 queue (i.e., the server is always available). ■

In the following discussion, we first derive $f(\cdot)$, $\alpha(\cdot)$, and $\pi(\cdot)$ for the general discrete time gated system with vacations where the gate closes when the server begins a vacation.

2.1 Derivation of $\pi(\cdot)$

Let b_k denote the probability that a service takes k slots of time, k=0, 1, 2,…, and let $\beta(z)$ denote its z-transform. By definition,

$$\beta(z) \equiv \sum_{k=0}^{\infty} b_k z^k. \tag{2}$$

Let P_j denote the probability that j customers arrive during a service. Then,

$$P_j = \sum_{k=j}^{\infty} \binom{k}{j} p^j (1-p)^{k-j} b_k, \qquad j=0,\ 1,\ 2,\ \cdots. \tag{3}$$

Note that the term p in (3) is the parameter for a Bernoulli arrival process.

If we now define the probability generating function of the number of customer arrivals during a service period as

$$h(z) \equiv \sum_{j=0}^{\infty} p_j z^j, \tag{4}$$

then, it is easily seen that

$$\pi(z) = \frac{(1-\rho)(z-1)h(z)}{z-h(z)}, \tag{5}$$

where $\rho = p\beta'(1)$ (see[1], pp. 212-216, for an analogous result in continuous time).

Substituting (3) into (4), we have

$$h(z) = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} p^j (1-p)^{k-j} b_k z^j = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (pz)^j (1-p)^{k-j} b_k$$

$$= \sum_{k=0}^{\infty} b_k (1-p+pz)^k = \beta(1-p+pz). \tag{6}$$

Substitution of (6) into (5) gives

$$\pi(z) = \frac{(1-\rho)(z-1)\beta(1-p+pz)}{z-\beta(1-p+pz)}. \tag{7}$$

2.2 Derivation of $\alpha(\cdot)$

Let v_k denote the probability that a vacation takes k slots of time, k=0, 1, 2…, and let $\bar{\omega}$ (z) denote its z-transform. By definition,

$$\bar{\omega} \equiv \sum_{k=0}^{\infty} v_k z^k. \tag{8}$$

Let Q_j denote the probability that j customers arrive during a vacation. Then,

$$Q_j = \sum_{k=j}^{\infty} \binom{k}{j} p^j (1-p)^{k-j} v_k, \qquad j=0,\ 1,\ 2\cdots. \tag{9}$$

Therefore,

$$\alpha(z) \equiv \sum_{j=0}^{\infty} Q_j z^j = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j} p^j(1-p)^{k-j}v_k z^j = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(pz)^j(1-p)^{k-j}v_k$$

$$= \sum_{k=0}^{\infty}(1-p+pz)^k v_k = \bar{\omega}(1-p+pz). \quad (10)$$

2.3 Derivation of f(·)

The term f(·) is derived as follows. Note that the gate closes when the server begins a vacation. Let N denote the number of customers present in the system when the vacation begins. Then,

$$P\{N=j\} = \sum_{k=0}^{\infty} P\{N=k\}\ A\{j\,|\,V+B^{*(k)}\}, \quad (11)$$

where $A\{j\,|\,V+B^{*(k)}\}$ is the probability of j arrivals during {a vacation followed by the sum of k service periods}. Note that $B^{*(k)}$ is the convolution of k independent services. Also note that V and $B^{*(k)}$ are independent.

Let $b_j^{(k)}$ denote the probability that a k-fold service period takes j slots of time. Also let $\beta^{(k)}(z)$ denote its z-transform. Then, since the service times are independent, we have

$$\beta^{(k)}(z) \equiv \sum_{j=0}^{\infty} b_j^{(k)} z^j = \{\beta(z)\}^k. \quad (12)$$

Let $A\{j\,|\,m, n\}$ denote the probability of j arrivals during a vacation of length m (units in slots of time) followed by a service period of length n. Then, conditioning on the number of arrivals during the vacation, we obtain

$$A\{j\,|\,m, n\} = \sum_{i=0}^{m}\binom{m}{i} p^i(1-p)^{m-i}\binom{n}{j\text{-}1} p^{j-i}(1-p)^{n-(j-i)}. \quad (13)$$

Thus,

$$A\{j\,|\,V+B^{*(k)}\} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A\{j\,|\,m, n\} v_m b_n^{(k)}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\binom{m}{i} p^i(1-p)^{m-i}\binom{n}{j\text{-}1} p^{j-i}(1-p)^{n-(j-i)} v_m b_n^{(k)}. \quad (14)$$

From (11) and (13), we obtain

$$f(z) = \sum_{j=0}^{\infty} P\{N=j\}z^j = P\{N=k\}\ A\{j\,|\,V+B^{*(k)}\}z^j$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i} p^i(1-p)^{m-i}\binom{n}{j\text{-}1} p^{j-i}(1-p)^{n-(j-i)} v_m b_n^{(k)}\right\} z^j.$$

After some modest algebra, the above equation reduces to

$$f(z) = f\{\beta(1-p+pz)\}\ \bar{\omega}(1-p+pz). \quad (15)$$

Refer to <Appendix> for the detailed derivation of equation (15). Note here that $\beta(1-p+pz)$ is the z-transform of the number of arrivals during a service period and $\overline{\omega}(1-p+pz)$ is that during a vacation period.

2.4 Derivation of the residence time distribution

In equations (7), (10), and (15), we derived the discrete time version of $f(\cdot)$, $\alpha(\cdot)$, and $\pi(\cdot)$ in terms of $\beta(\cdot)$ and $\overline{\omega}(\cdot)$. It is interesting to note that the term $(1-p+pz)$ in $f(\cdot)$, $\alpha(\cdot)$, and $\pi(\cdot)$ is replaced by $(\lambda-\lambda s)$ in the well known continuous time case where the server closes the gate when it returns from a vacation.

Now we let $W(\cdot)$ denote the distribution function of the residece time (queueing time + service time) of an arriving customer in a vacation system with a FCFS queueing discipline, and let $\tilde{W}(\cdot)$ denote the z-transform of $W(\cdot)$. Let $W_1(\cdot)$ and $\tilde{W}_1(\cdot)$ denote the analogous functions for the corresponding standard B/G/1 queueing system (the system without vacations).

Remark 1 Note that $g(\cdot)$ and $\pi(\cdot)$ are also the p.g.f. for the number of customers present in a vacation system, and in the corresponding standard system *at a customer arrival epoch.* This alternate interpretation follows from a theorem by Burke(see [1], pp. 187).

Remark 2 Note that under a FCFS queueing discipline, the customers that a departing customer leaves behind are precisely those customers who arrived during the departing customer's residence time.

Thus, if we let w_k, $k=0, 1,\cdots$, denote the probability mass function of the residence time in the corresponding standard system, then

$$\pi(z) = \sum_{j=0}^{\infty}\left[\sum_{k=j}^{\infty}\binom{k}{j} p^j(1-p)^{k-j}\, w_k\right] z^j = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(pz)^j(1-p)^{k-j}w_k$$

$$= \sum_{k=0}^{\infty} w_k(1-p+pz)^k = \tilde{W}_1(1-p+pz). \qquad (16)$$

It is easily seen that the same argument holds for the vacation system. That is,

$$g(z) = \tilde{W}(1-p+pz). \qquad (17)$$

Substituting (16) and (17) into (1), we get

$$\tilde{W}(1-p+pz) = f(z)\frac{1-\alpha(z)}{\alpha'(1)(1-z)}\tilde{W}_1(1-p+pz), \text{ or} \qquad (18a)$$

$$\tilde{W}(z) = f\left(\frac{z-(1-p)}{p}\right)\frac{1-\alpha\left(\frac{z-(1-p)}{p}\right)}{\alpha'(1)\left(1-\frac{z-(1-p)}{p}\right)}\tilde{W}_1(z). \qquad (18b)$$

We can now easily obtain the first and second moment of the residence time distribution from equation (18b).

3. Conclusion

In this note, we considered a discrete time B/G/1 queueing system with vacations, where the server closes the gate when it begins a vacation. We obtained the distribution of the number of customers at a random point in time, and in turn, the distribution of the residence time for a customer. It is noted that solutions for our discrete time B/G/1 gated vacation model are analogous to those for the continuous time M/G/1 gated vacation model. That is, the term $(\lambda-\lambda s)$ in the continuous case is replaced by $(1-p+pz)$ in the discrete case. It is also noted that two different gate-closing policies(where the gate closes when the server *ends* a vacation vs. when it *begins* a vacation) make no difference in performance measures.

<Appendix> (Derivation of equation (15))

$$f(z) = \sum_{j=0}^{\infty} P\{N=j\}z^j = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P\{N=k\}\ A\{j\ ;\ V+B^{*(k)}\}z^j$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\ p^i(1-p)^{m-i}\binom{n}{j\text{-}1}p^{j-i}(1-p)^{n-(j-i)}v_m b_n^{(k)}\right\}z^j$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{m+n} z^j\sum_{i=0}^{m}\binom{m}{i}\ p^i(1-p)^{m-i}\binom{n}{j\text{-}1}p^{j-i}(1-p)^{n-(j-i)}v_m b_n^{(k)}\right\}$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\ (pz)^i(1-p)^{m-i}\sum_{j=1}^{i+n}\binom{n}{j\text{-}1}(pz)^{j-i}(1-p)^{n-(j-i)}v_m b_n^{(k)}\right\}$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\ (pz)^i(1-p)^{m-i}\sum_{l=0}^{n}\binom{n}{l}\ (pz)^l(1-p)^{n-l}v_m b_n^{(k)}\right\}$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty} v_m\sum_{i=0}^{\infty}\binom{m}{i}\ (pz)^i(1-p)^{m-i}\right\}\left\{\sum_{n=0}^{\infty} b_n^{(k)}\sum_{l=0}^{n}\binom{n}{l}\ (pz)^l(1-p)^{n-l}\right\}$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\left\{\sum_{m=0}^{\infty} v_m\ (1-p+pz)^m\right\}\left\{\sum_{n=0}^{\infty} b_n^{(k)}\ (1-p+pz)^n\right\}$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\ \bar{\omega}(1-p+pz)\beta^{(k)}(1-p+pz)$$

$$= \sum_{k=0}^{\infty} P\{N=k\}\ \{\beta(1-p+pz)\}^k\ \bar{\omega}(1-p+pz)$$

$$= f(\beta(1-p+pz))\bar{\omega}(1-p+pz).$$

■

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