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LAWS OF THE ITERATED LOGARITHM FOR SYMMETRIC LÉVY PROCESSES

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1. Introduction

Let $\{X_t, t \geq 0\}$ be a one-dimensional symmetric Lévy process without Gaussian component. The characteristic function is expressed as follows;

$$E\exp(iuX_t) = \exp(-t\psi(u)),$$

where

$$\psi(u) = 2\int_0^\infty (1 - \cos ux)\nu(dx).$$

Here ν is a measure on $\mathcal{R} - \{0\}$ satisfying $\int(1 \wedge x^2)\nu(dx) < \infty$. Throughout this work, we assume that

$$\int_0^\infty \frac{1}{\lambda + \psi(u)}\, du < \infty \quad \text{for some} \quad \lambda > 0 \tag{1.1}$$

and

$$\int_0^1 x\ \nu(dx) = \infty. \tag{1.2}$$

It is well-known that under (1.1) and(1.2), there exists a version of local time $L(t, x, w)$ which is jointly measurable in (t, x, w) and satisfies occupation time density formula. Furthermore, we assume that

(1.3) $\quad \psi(u)$ is regularly varying at 0 of order α, $\quad 1 < \alpha < 2$.

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We also need to have a.s. joint continuity of $L(t,x)$ in (t,x), for which the necessary and sufficient condition was obtained in [1] and [2]. We need to introduce some notation. Define

$$\phi^2(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta x}{\psi(\theta)}d\theta$$

and $\overline{\phi}$ to be the monotone rearrangement of ϕ. Let

$$I(\overline{\phi}) = \int_0^1 \frac{\overline{\phi}(x)}{x(\log 1/x)^{1/2}}\ dx\ .$$

Assume that

$$(1.4) \qquad\qquad I(\overline{\phi}) < \infty\ .$$

In [1] and [2], under (1.1) and (1.2), (1.4) was shown to be equivalent to joint continuity of $L(t,x)$. The main purpose of this work is to obtain the exact law of iterated logarithm(LIL) of lim sup type for maximal local time, $L^*(t) = \sup_x L(t,x)$. For Brownian Motion, Kesten [6] showed that

$$\limsup_{t\to\infty}\frac{L(t,0)}{\sqrt{2t\ llt}} = \limsup_{t\to\infty}\frac{L^*(t)}{\sqrt{2t\ llt}} = 1 \quad \text{a.s.}$$

using Ray-Knight theorem, where llt denotes $\log\log t$. For symmetric stable process of index α, $1<\alpha<2$. Donsker and Varadhan [4] obtained the analogue using the large deviation theory. Recently, Marcus and Rosen [7] proved the exact lim sup result for $L(t,0)$ for the processes considered in this work. They obtained

$$\limsup_{t\to\infty}\frac{L(t,0)}{t\psi^{-1}(llt/t)} = C_1(\alpha) \quad \text{a.s.}$$

where

$$(C_1(\alpha))^\alpha = (C(\alpha))^{-1}$$

and

$$C(\alpha) = (\alpha-1)^{\alpha-1}\left(\frac{\pi}{\Gamma(1-1/\alpha)\Gamma(1/\alpha)}\right)^\alpha\ .$$

Our main result is to obtain the sharp lim sup result for $L^*(t)$ with the exactly same rate of growth as for $L(t,0)$. In fact, this is not unexpected because it was proved in [9] that the tail probabilities for $L^*(t)$ and $L(t,0)$ have the same order of exponential decay. As a result of LIL of lim sup type for $L^*(t)$, we obtain LIL of lim inf type for the Lebesgue measure of the range up to time t and $\sup_{s\leq t}|X_s|$. Although it is not possible to obtain the sharp results, it is interesting to find that they have the same rate of growth, when they are small. To be more precise, let

$$A_t = \sup_{s\leq t}|X_s| \ ,$$
$$R_t = \{x : X_s = x, \quad s \leq t\}$$

and $m(R_t)$ be its Lebesgue measure. Then we prove that

$$A_t \approx m(R_t) \approx \frac{1}{\psi^{-1}(llt/t)} \quad a.s.$$

when they are small, where "$\approx$" means that the ratio of both sides is bounded above and below by some finite positive constants. Furthermore,we prove the lim inf LIL for maximum of $|X|$ when its large jumps are removed. That is, define

$$X_t^1(a) = X(t) - \sum_{u\leq t}(X_u - X_{u-})\chi\{|X_u - X_{u-}| > a\}$$
$$A_t^1(a) = \sup_{s\leq t}|X_s^1(a)| \ .$$

Then it turns out that

$$A_t^1(a_t) \approx m(R_t) \approx A_t$$

when they are small for appropriate a_t, although for symmetric stable processes, it was proved that by Griffin [5]

$$A_t^1(a_t) \approx m(R_t) \ll A_t$$

when they are large.

2. Main Results

In this section, we prove lim sup LIL for $L^*(t)$ and lim inf LIL for A_t, $m(R_t)$ and $A_t^1(a_t)$. Throughout this section, we assume that (1.1), (1.2), (1.3) and (1.4) hold. The basic probability estimate for $L^*(t)$ is provided by [9].

LEMMA 1[9]. *Suppose*

$$\lim_{t\to\infty} t/\lambda_t = \lim_{t\to\infty} \psi(\lambda_t/t) = \infty$$

and

$$\limsup_{t\to\infty} \frac{ll\lambda_t}{t\psi(\lambda_t/t)} < \infty.$$

Then

$$-\log P(L^*(t) > \lambda_t) \sim C(\alpha)t\psi(\lambda_t/t)$$

where $C(\alpha) = (\alpha-1)^{\alpha-1}\left(\frac{\pi}{\Gamma(1-1/\alpha)\Gamma(1/\alpha)}\right)^{\alpha}$ *and " $\sim$ " means that the ratio tends to 1 as* $t \to \infty$.

THEOREM 2.1.

$$\limsup_{t\to\infty} \frac{L^*(t)}{t\psi^{-1}(llt/t)} = C_1(\alpha) \quad a.s.$$

Proof. Let $h(t) = t\psi^{-1}(llt/t)$. To obtain the upper bound, let $t_k = \theta^k, \theta > 1$. For given $\epsilon > 0$, choose $\delta > 0$ such that $(1-\delta)(1+\epsilon)^\alpha > 1$. Lemma 1 implies that for k sufficiently large,

$$\begin{aligned} P\Big(L^*(t_k) &> (1+\epsilon)C_1(\alpha)h(t_{k-1})\Big) \\ &\le \exp\left(-(1-\delta)(1+\epsilon)^\alpha llt_{k-1}/\theta^{\alpha-1}\right) \\ &= \Big((k-1)\log\theta\Big)^{-(1-\delta)(1+\epsilon)^\alpha/\theta^{\alpha-1}}. \end{aligned}$$

Choose $\theta > 1$ so that $(1-\delta)(1+\epsilon)^\alpha > \theta^{\alpha-1}$. Then by the Borel-Cantelli Lemma, we have for any $\epsilon > 0$

$$\limsup_{t\to\infty} \frac{L^*(t)}{h(t)} \le (1+\epsilon)C_1(\alpha) \qquad a.s.$$

To prove the lower bound result, let $t_k = \exp(k^\beta)$, $\beta > 1$. It suffices to prove divergence of

$$\sum_k P\Big((L(t_k) - L(t_{k-1}))^* > (1-\epsilon)C_1(\alpha)h(t_k)\Big)$$

for any $\epsilon > 0$, where

$$\Big((L(t_k) - L(t_{k-1})\Big)^* = \sup_x \Big(L(t_k, x) - L(t_{k-1}, x)\Big).$$

For given $\epsilon > 0$, choose $\delta > 0, \eta > 0, \beta > 1$ so that $\beta(1+\delta)(1+\eta)(1-\epsilon)^\alpha < 1$. Again by Lemma 1, we have

$$\begin{aligned} P\Big((L(t_k) - L(t_{k-1}))^* &> (1-\epsilon)C_1(\alpha)h(t_k)\Big) \\ &\geq \exp\Big(-(1+\delta)(1+\eta)(1-\epsilon)^\alpha llt_k\Big) \\ &= k^{-\beta(1+\delta)(1+\eta)(1-\epsilon)^\alpha} \end{aligned}$$

for k sufficiently large.

Now to prove the lim inf results for $A_t, m(R_t)$ and $A_t^1(a_t)$, we need to have probability estimates for $|X_t|$.

LEMMA 2.
(1) *For $K_1 > 0$, and s sufficiently large,*

$$P\left(|X_s| \leq \frac{K_1}{\psi^{-1}(1/s)}\right) \geq \frac{2}{\pi}\exp\left(-(\frac{\pi}{2K_1})^\alpha\right).$$

(2) *For $0 < \delta < \frac{\pi}{2}$, $K_2 > 0$, and s sufficiently large,*

$$P\left(|X_s| \leq \frac{K_2}{\psi^{-1}(1/s)}\right) \geq \frac{4\delta}{\pi} + \frac{2}{\pi\alpha}(\frac{K_2}{\delta})^\alpha.$$

Proof. For the lower bound, let $\lambda_1 = K_1/\psi^{-1}(1/s)$, and use the inversion formula:

$$\begin{aligned}
P(|X_s| \le \lambda_1) &= \frac{2}{\pi}\int_0^\infty \frac{\sin u\lambda_1}{u}\exp(-s\psi(u))\,du \\
&\ge \frac{4\lambda_1}{\pi^2}\int_{0<u\lambda_1<\frac{\pi}{2}} \exp(-s\psi(u))\,du \\
&\ge \frac{2}{\pi}\exp\left(-s\psi(\frac{\pi}{2\lambda_1})\right) \\
&\ge \frac{2}{\pi}\exp\left(-(\frac{\pi}{2K_1})^\alpha\right).
\end{aligned}$$

For the upper bound, let $\lambda_2 = K_2/\psi^{-1}(1/s)$. Then

$$\begin{aligned}
P(|X_s| \le \lambda_2) &\le \frac{4\lambda_2}{\pi}\int_{0<u\lambda_2<\delta} \exp(-s\psi(u))\,du \\
&\quad + \frac{2}{\pi}\int_{u\lambda_2\ge\delta} \frac{1}{u}\exp(-s\psi(u))\,du \\
&\le \frac{4\delta}{\pi} + \frac{2}{\pi s}\int_{u\lambda_2\ge\delta}\frac{1}{u\psi(u)}du \\
&\sim \frac{4\delta}{\pi} + \frac{2}{\pi\alpha s\psi(\delta/\lambda_2)} \\
&\sim \frac{4\delta}{\pi} + \frac{2}{\pi\alpha}\left(\frac{K_2}{\delta}\right)^\alpha
\end{aligned}$$

where the regular variation of ψ at zero is used in the last two steps. See [3].

Finally, we prove the three lim inf results.

THEOREM 2.2. *There exist positive finite constants* C_2, C_3, and C_4 *such that*

$$\liminf_{t\to\infty} m(R_t)\,\psi^{-1}(llt/t) = C_2 \quad a.s. \tag{2.1}$$

$$\liminf_{t\to\infty} A_t\,\psi^{-1}(llt/t) = C_3 \quad a.s. \tag{2.2}$$

$$\liminf_{t\to\infty} A_t^1(a_t)\,\psi^{-1}(llt/t) = C_4 \quad a.s. \tag{2.3}$$

where $a_t = \left(\psi^{-1}(llt/t)\right)^{-1}$.

Proof. Since $t \le L^*(t)m(R_t)$, we have

$$\frac{1}{C_1(\alpha)} = \liminf_{t\to\infty} \frac{t\psi^{-1}(llt/t)}{L^*(t)} \le \liminf_{t\to\infty} m(R_t)\psi^{-1}(llt/t) \quad a.s.$$

To prove (2.1) and (2.2), it remains to show that for some $C > 0$,

$$\liminf_{t\to\infty} A_t\,\psi^{-1}(llt/t) \le C \quad a.s.$$

Let $s = t/llt$, and choose K_1 large so that $\frac{4}{\pi}\exp\left(-(\pi/K_1)^\alpha\right) > 1$. Then we have

$$\begin{aligned} P\left(A_s \le \frac{K_1}{2\psi^{-1}(1/s)}\right) &\ge 2P\left(|X_s| \le \frac{K_1}{2\psi^{-1}(1/s)}\right) - 1 \\ &\ge \frac{4}{\pi}\exp\left(-(\pi/K_1)^\alpha\right) - 1. \end{aligned}$$

Let $\frac{4}{\pi}\exp\left(-(\pi/K_1)^\alpha\right) - 1 = 2e^{-\xi}$, $n = [llt]+1$, and $M = K_1/2\psi^{-1}(1/s)$. For $k = 1, 2, \ldots, n$, let

$$E_k = \left\{\sup_{0\le u\le s} |X_{(k-1)s+u} - X_{(k-1)s}| \le M,\ |X_{ks}| \le M\right\}.$$

Observe that

$$P\left(\cap_{k=1}^n E_k \mid \mathcal{F}_{(n-1)s}\right) = \Pi_{k=1}^{n-1}\chi_{E_k} P(E_n \mid X_{(n-1)s}) \quad a.s.$$

where $\mathcal{F}_t$ is the smallest σ-*field* generated by $\{X_u,\ u \le t\}$. On the event E_{n-1},

$$\begin{aligned} P(E_n \mid X_{(n-1)s}) &\ge P(A_s \le M,\ -M \le X_s \le 0) \\ &= \frac{1}{2}P(A_s \le M) \end{aligned}$$

if $0 \le X_{(n-1)s} \le M$, and similarly if $-M \le X_{(n-1)s} \le 0$. By taking iterated conditional expectations, we have

$$\begin{aligned} P\left(A_t \le \frac{K_1}{\psi^{-1}(1/s)}\right) &\ge \left(\frac{1}{2}P(A_s \le M)\right)^n \\ &\ge (e \log t)^{-\xi}. \end{aligned}$$

The rest of the proof runs similarly to the proof of Theorem 3.3 in [8]. Finally, for the lim inf result for $A_t^1(a_t)$, observe that for $\epsilon > 0$, and t large

(2.4)
$$P(|X^1_{t/llt}(a_t)| \le \lambda) \ge P(|X_{t/llt}| \le \lambda)$$

(2.5)
$$\begin{aligned} P(|X^1_{t/llt}(a_t)| \le \lambda) &\le \exp(tG(a_t)/llt) P(|X_{t/llt}| \le \lambda) \\ &\le \exp(C_5(\alpha)(1+\epsilon)) P(|X_{t/llt}| \le \lambda) \end{aligned}$$

since
$$G(a) = \int_{|x|>a} \nu(dx) \sim C_5(\alpha)\psi(1/a) \text{ as } a \to \infty$$

by regular variation of ψ at zero where $C_5(\alpha) = \frac{\pi}{2\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}$. See Lemma 2.4 in [9]. (2.4) and Lemma 2.(1) yield the upper bound result for $A_t^1(a_t)$ by using the same technique for A_t. For the lower bound, we need to prove that for some $C > 0$, $\sum P\left(A^1_{t_k}(a_{t_k}) \le C a_{t_{k+1}}\right)$ converges for $t_k = 2^k$, which is fairly routine using Lemma 2.(2) and (2.5).

References

1. M. T. Barlow and J. Hawkes, *Application de l'entropie métrique á la continuité des temps locaux des processus de Lévy*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 237–239.
2. M. T. Barlow, *Necessary and sufficient condition for the continuity of local times of Lévy processes*, Ann. Probab. **16** (1988), 1389–1427.
3. N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987.
4. M. D. Donsker and S. R. S. Varadhan, *On laws of the iterated logarithm for local times*, Comm. Pure Appl. Math. **30** (1977), 707–753.

5. P. S. Griffin, *Laws of the iterated logarithm for symmetric stable processes*, Z. Wahrsch. verw. Gebeiete 68 (1985), 271–285.
6. H. Kesten, *An iterated logarithm law for local time*, Duke Math. J. **32** (1965), 447–456.
7. M. B. Marcus and Rosen, Jay., *Laws of the iterated logarithm for the local times of symmetric Lévy processes and recurrent random walks*, Preprint.(To appear in Ann. Probab.).
8. I-S. Wee, *Lower functions for processes with stationary independent increments* Probab. Theory Related. Fields **77** (1988), 551–566.
9. ———, *Tail probability estimate for maximum local time of Lévy processes*, (preprint).

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