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ON SELF-SIMILAR STOCHASTIC INTEGRAL PROCESSES

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1. Introduction

A stochastic process $X = \{X(t) : t \in T\}$, with an index set T, is said to be infinitely divisible (ID) if its finite dimensional distributions are all ID. An ID process X is said to be a stochastic integral process if $X = \{X(t) : t \in T\} \stackrel{\mathcal{D}}{=} \{\int f_t d\Lambda : t \in T\}$ where $f : T \times S \to R$ is a deterministic function and Λ is an ID random measure on a δ-ring $\mathcal{S}$ of subsets of an arbitrary non-empty set S with the property; there exists an increasing sequence $\{S_n\}$ of sets in $\mathcal{S}$ with $\cup_n S_n = S$. Here $\stackrel{\mathcal{D}}{=}$ denotes equality in all finite dimensional distributions.

Let $M_\alpha (0 < \alpha \leq 2)$ be a symmetric α-stable random measure on measurable space $(S, \mathcal{S})$ with a control measure m and $I(f) = \int_S f(s) dM_\alpha$ for all measurable function $f : S \to R$ satisfies the condition

$$\int_S |f_t(s)|^\alpha \, m(ds) < \infty \text{ for each } t.$$

Then $\{\int_S f_t(s) dM_\alpha : t \in T\}$ is symmetric α-stable process. [4, 8] gives us a Lepage representation of symmetric α-stable process and a characterization of self-similar α-stable process.

We consider integrals with respect to ID random measure which generalizes the stable random measure and are mostly interested in characteristic functions, series representation and self-similarities of stochastic integral process $\{X(t) : t \in T\} \stackrel{\mathcal{D}}{=} \{\int f_t(s) d\Lambda : t \in T\}$.

First, series representations involving arrival times in a Poisson process have been given by [2], for real independent increment processes without Gaussian components and with positive jumps. Series representation derived as a special case of a generalized shot noise generalizes

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various Lepage type representations of stochastic integral processes. (See [3, 5]).

Chapter 2 is to review technical back-ground such as identification of the space of Λ-integrable functions as a certain Musielak-Orlicz spaces L_{Φ_p} and characteristic functions of Λ-integrable functions in terms of certain parameters of an ID random measure.

Chapter 3 gives us a characterizations and necessary conditions of self-similar stochastic integral processes $\{\int f_t d\Lambda : t \in T\}$ where $f_t \in L_{\Phi_p}$.

DEFINITION 1.1. A random measure Λ is said to be an independently scattered ID random measure (for short, ID random measure) if for each $A \in \mathcal{S}, \Lambda(A)$ is ID random variable, and Λ is independently scattered random measure.

DEFINITION 1.2. A positive Borel measure on $R\backslash\{0\}$ is Lévy measure if it integrates the function $\min\{1, x^2\}$.

DEFINITION 1.3. A σ-finite measure ν on $\sigma(\mathcal{S})$ (= the smallest σ - field generated by $\mathcal{S}$) is said to be a control measure of the random measure Λ if Λ and ν have the same families of zero sets.

DEFINITION 1.4. A measurable function $f : (S, \sigma(\mathcal{S})) \to (R, \mathcal{B}(R))$ is said to be Λ-integrable if there exists a sequence $\{f_n\}$ of simple function such that

(1) $f_n \to f \quad \nu$-a.s.
(2) for every $A \in \sigma(\mathcal{S})$, the sequence $\{\int_A f_n d\Lambda\}$ converges in probability, as $n \to \infty$.

If f is Λ-integrable, then we put

$$\int_A f d\Lambda = p - \lim_{n\to\infty} \int_A f_n d\Lambda,$$

where $\{f_n\}$ satisfies (1) and (2).

DEFINITION 1.5. A stochastic process $\{X(t); t \in T\}$ is said to be self-similar with index $H(H - ss)$ if for some $H \in R$.

$$X(ct) \stackrel{\mathcal{D}}{=} c^H X(t) \text{ for any } c > 0.$$

2. $\wedge$-integrable functions and Musielak-Orlicz space

We assume $\Lambda(A)$ is an ID random variable for $A \in \mathcal{S}$. Then its characteristic function can be written in the Lévy-Khintchin form

$$(2.1)\quad \mathcal{L}(\wedge(A))^{\wedge}(u) = \exp\{iu\nu_0(A) - \frac{1}{2}u^2\nu_1(A) + \int_R \{e^{iux} - 1 - iuxI_{(|x|\le 1)}\}F_A(dx)\},$$

where $-\infty < \nu_0(A) < \infty, 0 \le \nu_1(A) < \infty, F_A$ is a Lévy measure on R and I_B is a indicator function of B.

We know that there is one-to-one correspondence between the class of ID random measures $\wedge(\cdot)$ on one hand and the class of parameters $\nu_o(\cdot), \nu_1(\cdot)$ and $F_{(\cdot)}$ on the other.

The following lemma introduces an explicit form of a control measure for a general ID random measure.

LEMMA 2.1. *Let ν_0, ν_1 and $F.$ be as in* (2.1) *and define*

$$\nu(A) = |\nu_0|(A) + \nu_1(A) + \int_R \min\{1, x^2\}\, F_A(dx)\ , A \in \mathcal{S}$$

Then $\nu : \sigma(\mathcal{S}) \to [0, \infty]$ is a control measure of $\wedge$.

Proof. Let $A_n \in \mathcal{S}, A_n \downarrow \phi$. Since $\wedge(A_n) \to 0$ in probability, we have that

$$\nu_0(A_n) \to 0, \nu_1(A_n) \to 0 \text{ and } \int_R \min\{1, x^2\}F_{A_n}(dx) \to 0.$$

We get $\nu(A_n) \to 0$ proving that ν is countably additive. Since $\nu(S_n) < \infty$, where $S_n \in \mathcal{S}$ is increasing and $\cup_n S_n = S$, we may uniquely extend ν to a σ-finite measure on $(S, \sigma(\mathcal{S}))$.

To prove that the σ-finite measure ν is a control measure of $\wedge$, let $A \in \sigma(\mathcal{S})$ be a $\wedge$-zero set. Then, for any $A_1 \in \mathcal{S}$ satisfying $A_1 \subset A, \wedge(A_1) = 0$ a.s. Decompose $A_1 = A_1' \cup A_1''$ such that $\nu_0(A_1') = {\nu_0}^+(A_1)$ and $\nu_0(A_1'') = -{\nu_0}^-(A_1)$. Since $\wedge(A_1') = \wedge(A_1'') = 0$, we get that $\nu_0({A_1}') = \nu_0({A_1}'') = 0$. Hence $|\nu_0|(A_1) = 0$.

We know that $\nu_1(A_1) = 0$ and $\int_R \min\{1, x^2\}F_{A_1}(dx) = 0$. Thus $\nu(A) = 0$ by an approximation theorem. ([1, Theorem 11.4]).

Conversely, let $A \in \sigma(\mathcal{S})$ be a ν-zero set. Then $\nu(A_1) = 0$ for any $A_1 \subset A, A_1 \in \mathcal{S}$. Therefore, $|\nu_0(A_1)| \leq |\nu_0|(A_1) = 0, \nu_1(A_1) = 0$ and $\int_R \min\{1, x^2\} F_{A_1}(dx) = 0$, i.e., $\wedge(A_1) = 0$ a.s. Let λ be an arbitrary but fixed control measure of $\wedge$.

LEMMA 2.2. *Let $F.$ be as in (2.1). Then there exists a unique σ-finite measure F on $\sigma(\mathcal{S}) \times \mathcal{B}(R)$ such that*

$$F(A \times B) = F_A(B), \text{ for all } A \in \mathcal{S}, B \in \mathcal{B}(R) \tag{2.2}$$

Moreover, there exists a function $Q : S \times \mathcal{B}(R) \to [0, \infty]$ such that

(1) *$Q(s, \cdot)$ is a Borel measure on $\mathcal{B}(R)$, for every $s \in S$.*
(2) *$Q(\cdot, B)$ is a $\sigma(\mathcal{S})$ - measurable function, for every $B \in \mathcal{B}(R)$.*
(3) *$\int_{S \times R} h(s, x) F(ds, dx) = \int_S \left[\int_R h(s, x) Q(s, dx)\right] \lambda(ds)$ for every $\sigma(S) \times \mathcal{B}(R)$-measurable function $h : S \times R \to [0, \infty]$.*
(4) *$\int_R \min\{1, x^2\} Q(s, dx) < \infty$, for every $s \in S$,*
(5) *$\lambda\{s \in S : a(s) = \sigma^2(s) = Q(s, R) = 0\} = 0$.*
(6) *$\mathcal{L}(\wedge(A))^\wedge(u) = \exp\{\int_A K(u, s)\lambda(ds)\}$ where $K(u, s) = iua(s) - \frac{1}{2}u^2\sigma^2(s) + \int_R\{e^{iux} - 1 - iuxI_{(|x| \leq 1)}\}Q(s, dx)$,*
$a(s) = \dfrac{d\nu_0}{d\lambda}(s), \sigma^2(s) = \dfrac{d\nu_1}{d\lambda}(s)$

Proof. By [7, Lemma 2.3], there exists a unique σ-finite measure F on $\sigma(S) \times \mathcal{B}(R)$ satisfying (2.2), and we can find a function $\rho : S \times \mathcal{B}(R) \to [0, \infty]$ such that

(a) $\rho(s, \cdot)$ is a Lévy measure on $\mathcal{B}(R)$, for every $s \in S$,
(b) $\rho(\cdot, B)$ is a $\sigma(\mathcal{S})$ - measurable function, for every $B \in \mathcal{B}(R)$.
(c) $\int_{S \times R} h(s, x) F(ds, dx) = \int_S \left[\int_R h(s, x) \rho(s, dx)\right] \lambda(ds)$ for every $\sigma(S) \times \mathcal{B}(R)$-measurable function $h : S \times R \to [0, \infty]$.

Since λ and ν are equivalent σ-finite measure on $\sigma(\mathcal{S})$ there exists a strictly positive and finite version of ψ of Radon-Nikodym derivative $\dfrac{d\nu}{d\lambda}$. Put

$$Q(s, dx) = \psi(s)\rho(s, dx).$$

Then (1), (2), (3) and (6) follow because

$$F_A(B) = F(A \times B) = \int_A \int_R I_B(x) Q(s, dx) \lambda(ds).$$

Since $\rho(s,\cdot)$ is a Lévy measure, (4) is satisfied.
Finally, note that $A_0 = \{s : a(s) = \sigma^2(s) = Q(s,R) = 0\}$ is a Λ-zero set, so that $\lambda(A_0) = 0$.

The following provides a necessary and sufficient condition for the existence of $\int_S f d\Lambda$ in terms of the deterministic characteristic of Λ.

THEOREM 2.3. *Let $f : S \to R$ be a $\sigma(\mathcal{S})$-measurable function. Then f is Λ-integrable if and only if the following three conditions hold;*

(1) $\int_S |U(f(s),s)|\lambda(ds) < \infty$,
(2) $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$ *and*
(3) $\int_S V_0(f(s),s)\lambda(ds) < \infty$,

where

$$U(u,s) = ua(s) + \int_R \{xuI_{(|xu|\le 1)} - uxI_{(|x|\le 1)}\}Q(s,dx),$$
$$V_0(u,s) = \int_R \min\{1, |ux|^2\}Q(s,dx)$$

Proof. See [7, Theorem 2.7]

We shall define a certain Musielak-Orlicz space and identify the set of Λ- integrable functions as Musielak-Orlicz space.
Let q be a nonnegative number such that

$$E|\Lambda(A)|^q < \infty \text{ for all } A \in \mathcal{S}.$$

For $0 \le p \le q$,
Define

$$\Phi_p(u,s) = U^*(u,s) + u^2\sigma^2(s) + V_p(u,s) \tag{2.3}$$

where

$$U^*(u,s) = \sup_{|c|\le 1}(|U(cu,s)|)$$
$$V_p(u,s) = \int_R \{|ux|^p I_{(|ux|>1)} + |ux|^2 I_{(|ux|\le 1)}\}Q(s,dx)$$

Then the following three conditions are satisfied

(1) For every $s \in S, \Phi_p(\cdot, s)$ is a continuous non-decreasing function on $[0, \infty)$ with $\Phi_p(o, s) = 0$

(2) $\lambda\{s : \Phi_p(u, s) = 0$ for some $u \neq 0\} = 0$

(3) There exists a numerical constant $C > 0$ such that $\Phi_p(2u, s) \leq C\Phi_p(u, s)$, for all $u \geq 0$ and $s \in S$.

Now, we can define the so-called Musielak-Orlicz space

$$L_{\Phi_p}(S; \lambda) = \{f \in L_0(S; \lambda) : \int_S \Phi_p(|f(s)|, s)\lambda(ds) < \infty\}.$$

The space $L_{\Phi_p}(S : \lambda)$ is a complete linear metric space with norm defined by

$$\|f\|_{\Phi_p} = \inf\{c > 0\, ; \int_S \Phi_p(c^{-1}|f(s)|, s)\lambda(ds) \leq c\}.$$

If the function Φ_p is independent of S, i.e. $\Phi_p(u, s) = \Phi_p(u)$, then the corresponding space $L_{\Phi_p}(S; \lambda)$ is called an Orlicz space.

THEOREM 2.4. *Let $0 \leq p \leq q$ and Φ_p be as in (2.3). Then*

$$\{f : f \text{ is } \wedge - \text{integrable and } E|\int f d\wedge|^p < \infty = L_{\Phi_p}(S : \lambda)$$

$p = 0$ signifies that $\{f : f$ is $\wedge -$ integrable $\} = L_{\Phi o}(S; \lambda)$.

Proof. See [7, Theorem 3.3].

Let $\lambda^{(1)}$ be a arbitrary probability measure on $(S, \sigma(\mathcal{S}))$ equivalent to λ.

Set

$$R(r, s) = \inf\{x > 0; Q(s, [-x, x]^c) \leq r\}, r > 0$$

$$R^{(1)}(r, s) = R(r\frac{d\lambda^{(1)}}{d\lambda}(s), s), r > 0,\ s \in S$$

where the version of the Radon-Nikodym derivative $\frac{d\lambda^{(1)}}{d\lambda}$ is chosen to be strictly positive and finite everywhere.

Let f be a $\wedge$-integrable function i.e., $f \in L_{\Phi_0}(S; \lambda)$. Define

$$F_f(A) = \int_S \int_R I_{A\setminus\{0\}}(xf(s))Q(s, dx)\lambda(ds) \tag{2.4}$$

Define $a_f = \int_S U(f(s), s)\lambda(ds), \quad {\sigma_f}^2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds)$.

THEOREM 2.5. *Assume $f \in L_{\Phi_0}(S;\lambda)$. Then F_f is Lévy measure and*

$$
\begin{aligned}
&\mathcal{L}\Big(\int f d\Lambda\Big)^{\wedge}(u)\\
&= \exp\Big\{iua_f - \frac{1}{2}u^2\sigma f^2 + \int_R \{e^{iux} - 1 - iuxI_{(|x|\le 1)}\}F_f(dx)\Big\}\\
&= \exp\Big\{iu\int_S a(s)f(s)\lambda(ds) - \frac{1}{2}\int_S |f(s)|^2\sigma^2(s)\lambda(ds)\\
&\quad + \int_R \{e^{iuR^{(1)}(r,s)f(s)} - 1 - iuR^{(1)}(r,s)f(s)I_{(R^{(1)}(r,s)\le 1)}\}\lambda^{(1)}(ds)dr\Big\}.
\end{aligned}
$$

Proof. F_f is Lévy measure, since we have

$$
\begin{aligned}
\int_{\{|x|\le 1\}} x^2 F_f(dx) &= \int_S \int_{\{|f(s)x|\le 1\}} |f(s)x|^2 Q(s,dx)\lambda(ds)\\
&\le \int_S \Phi_0(|f(s)|,s)\lambda(ds)\\
&< \infty,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\{|x|>1\}} F_f(dx) &= \int_S \int_{\{|f(s)x|>1\}} Q(s,dx)\lambda(ds)\\
&\le \int_S \Phi_0(|f(s)|,s)\lambda(ds)\\
&< \infty.
\end{aligned}
$$

Since, for every $x \ge 0$ and $s \in S$

$$Leb\{r > 0; R^{(1)}(r,s) > x\} = \frac{d\lambda}{d\lambda^{(1)}}(s)Q(s,[-x,x]^c)$$

We get,

$$F_f(A) = \int_0^\infty \int_S I_{A\setminus\{0\}}(R^{(1)}(r,s)f(s))\lambda^{(1)}(ds)dr. \tag{2.5}$$

By [7, Theorem 2.7] we complete the proof.

3. Self-similarity of stochastic integral processes

Stochastic integral processes $\{\int f_t d\Lambda; t \in T\}$ are always assumed to be real-valued and defined for $t \in T = [0, \infty)$. All stochastic processes will be discussed in terms of finite-dimensional distributions.
By $\{\int f_t d\Lambda; t \in T\} \stackrel{\mathcal{D}}{=} \{\int g_t d\Lambda; t \in T\}$ we mean the equality of all finite-dimensional distributions. $\int f_t d\Lambda \stackrel{d}{\sim} \int g_t d\Lambda$ means the equality of one-dimensional distributions for fixed t. Recall any stochastic process $\{\int f_t d\Lambda; t \in T\}$ is called H-ss,si(stationary increment) if, for every constant $c > 0, H \in R$,

$$\{\int f_{ct} d\Lambda; t \in T\} \stackrel{\mathcal{D}}{=} \{c^H \int f_t d\Lambda; t \in T\}$$

$$\{\int f_{t+h} d\,\Lambda - \int f_h d\Lambda; t \in T\} \stackrel{\mathcal{D}}{=} \{\int f_t d\,\Lambda - \int f_0 d\Lambda; t \in T\} \text{ for any } h > 0.$$

Let $\Lambda = \{\Lambda(A); A \in \mathcal{S}\}$ be a symmetric ID random measure without Gaussian component. Then the characteristic function of $\int f d\Lambda$ can be written in Lévy's form

$$\mathcal{L}(\int f d\Lambda)^{\wedge}(u) = \exp\{2 \int_0^\infty \{\cos ux - 1\} F_f(dx)\}, u \in R$$

where, $f \in L_{\Phi_0}(S; \lambda)$ and F_f is a symmetric Lévy measure.

THEOREM 3.1. *Assume that $f_t \in L_{\Phi_0}(S; \lambda)$ for each $t \in T$ and $\{\int f_t d\Lambda; t \in T\}$ is H-ss,si stochastic integral process,*

(1) *if $H < 0$, then $\|f_t\|_{\Phi_0} = 0$.*
(2) *if $f_1 \in L_{\Phi_1}$ and $H \neq 1$, then $f_t \in L_{\Phi_1}$ and $E \int f_t d\Lambda = 0$.*

Proof. (1) By H-ss, $\int f_0 d\Lambda = 0$ a.s. and

$$\int f_t \, d\Lambda \stackrel{d}{\sim} t^H \int f_1 d\Lambda \to 0 \text{ as } t \to \infty.$$

On the other hand, by si,

$$\int f_t d\Lambda = \int f_t d\,\Lambda - \int f_0 d\Lambda \stackrel{d}{\sim} \int f_{t+h} d\,\Lambda - \int f_h d\Lambda$$

which tends to 0 as $h \to \infty$. Hence $\int f_t d\Lambda = 0$ a.s. From the facts that $\int f_t d\Lambda = 0$ a.s. and $f_t \in L_{\Phi_0}$, there exists a sequence $\{f_{t,n}\}$ of simple functions such that $f_{t,n} \to f_t$ λ-a.s. and $\int f_{t,n} d\Lambda \to 0$ in probability as $n \to \infty$. Since Λ is symmetric, we have $a(s) = 0$ and $Q(s,\cdot)$ is symmetric. Thus $U(\cdot, s) \equiv 0$ λ - a.s. which implies

$$\begin{aligned}\int \Phi_0(|f_{t,n}(s)|, s)\lambda(ds) &= \int V_0(|f_{t,n}(s)|, s)\lambda(ds)\\ &= \int \min\{1, x^2\} F_{f_{t,n}}(dx) \to 0\end{aligned}$$

i.e.,

$$\|f_{t,n}\|_{\phi_0} \to 0, \text{ as } n \to \infty.$$

From the triangle inequality,we get $\|f_t\|_{\phi_0} = 0$.

Proof. (2) By $\int f_t d\Lambda \overset{d}{\sim} t^H \int f_1 d\Lambda$ and uniqueness of Lévy measure corresponding to same distribution functions, we have

$$\begin{aligned}\int \Phi_1(|f_t(s)|, s)\lambda(ds) &= \int \{|x| I_{(|x|>1)} + |x|^2 I_{(|x|\le 1)}\} F_{f_t}(dx)\\ &= \int \{|x| I_{(|x|>1)} + |x|^2 I_{(|x|\le 1)}\} F_{t^H f_1}(dx)\\ &= \int \Phi_1(|t^H f_1(s), s)\lambda(ds) < \infty,\end{aligned}$$

i.e., $f_t \in L_{\phi_1}$, which implies $E|\int f_t d\wedge| < \infty$.
By H-ss,si, it follows that

$$\begin{aligned}E[\int f_t d\Lambda] &= E[\int f_{2t} d\wedge - \int f_t d\Lambda]\\ &= (2^H - 1)E[\int f_t d\Lambda]\end{aligned}$$

so $(2^H - 2)E[\int f_t d\Lambda] = 0$. Since $H \neq 1$, we have $E[\int f_t d\Lambda] = 0$.

The following theorems gives us necessary conditions of H-ss stochastic integral process $\{\int f_t d\Lambda; t \in T\}$.

Let $\lambda^{(1)}$ be an arbitrary probability measure on $(S, \sigma(\mathcal{S}))$ equivalent to λ.

THEOREM 3.2. *Assume that* $f_1 \in L_{\Phi_0}(S;\lambda)$ *and* $f_t = t^H f_1$ λ-a.s. *for any* $t \in T$. *Then* $\{\int f_t d\Lambda; t \in T\}$ *is H-ss stochastic integral process.*

Proof. For any $t \in [0,\infty]$, obviously $f_t \in L_{\Phi_0}$ and $f_{ct} = c^H f_t$ $\lambda^{(1)}$-a.s. We have

$$\begin{aligned}\mathcal{L}(\int f_{ct} d\Lambda)^{\wedge}(u) &= \exp\{2\int_0^\infty \{\cos ux - 1\} F_{f_{ct}}(dx)\}\\ &= \exp\{2\int_0^\infty \int_S \{\cos(uR^{(1)}(r,s) f_{ct}(s)) - 1\}\lambda^{(1)}(ds)dr\}\\ &= \exp\{2\int_0^\infty \int_S \{\cos(uR^{(1)}(r,s) c^H f_t(s)) - 1\}\lambda^{(1)}(ds)dr\}\\ &= \exp\{2\int_0^\infty \{\cos ux - 1\} F_{c^H f_t}(dx)\}\end{aligned}$$

and $\sum a_j f_{ct_j} = c^H \sum_{j=1}^n a_j f_{t_j}$ $\lambda^{(1)}$-a.s for any $a_j \in R, t_j \in T$. Thus,

$$\sum_{j=1}^n a_j \int f_{ct_j} d\Lambda \overset{d}{\approx} c^H \sum_{j=1}^n a_j \int f_{t_j} d\Lambda .$$

Let $\{\xi_j\}, \{\Gamma_j\}$, and $\{\varepsilon_j\}$ be independent sequence of random variables such that $\{\xi_j\}$ is a sequence of i.i.d. random variables in $(S, \sigma(S))$ with $\mathcal{L}(\xi_j) = \lambda^{(1)}, \{\Gamma_j\}$ is j^{th} arrival time of a Poisson process N_t with parameter 1 i.e. $\Gamma_j = \inf\{t > 0; N_t = j\}$. and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variable with $P\{\varepsilon_n = -1\} = P\{\varepsilon_n = 1\} = \frac{1}{2}$. Put

$$X_n(t) = \sum_{j=1}^n \varepsilon_j R^{(1)}(\Gamma_j, \xi_j) f_t(\xi_j).$$

Under the same assumption as Theorem 3.2 , we know that $f_t \in L_{\phi_0}$. By [5, Proposition 2], $X_n(t)$ converges a.s. (to $X(t)$) and $\{X(t); t \in T\} \overset{\mathcal{D}}{=} \{\int f_t d\Lambda; t \in T\}$. Therefore, the following theorem holds.

THEOREM 3.3. *Assume that* $f_1 \in L_{\Phi_0}(S;\lambda)$ *and* $f_t = t^H f_1$ λ-a.s. *for any* $t \in T$ *Then*

(1) $X_n(t)$ *converges a.s (to* $X(t)$*),*
(2) $X(t) \overset{\mathcal{D}}{=} \int f_t d\Lambda$,
(3) $\{X(t) : t \in T\}$ *is H-ss stochastic process.*

From now on, we will assume that $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ is a ID random measure without Gaussian component and have the characteristic function as the following form

$$\mathcal{L}(\int f d\Lambda)^{\wedge}(u) = \exp\{\int_R \{e^{iux} - 1 - iux\} F_f(dx)\},$$

where $f \in L_{\Phi_p}, p \geq 1$, and F_f is defined in (2.4).

Put

$$Y_n(t) = \sum_{j=1}^{n} R^{(1)}(\Gamma_j, \xi_j) f_t(\xi_j) - C_{f_t}(\Gamma_n)$$

where $f_t \in L_{\phi_p}, p \geq 1$ and

$$C_{f_t}(a) = \int_0^a \int_S R^{(1)}(r, s) f_t(s) \lambda^{(1)}(ds) dr, a > 0.$$

THEOREM 3.4. *Assume that $f_1 \in L_{\Phi_p}(S; \lambda)$, $p \geq 1$, and $f_t = t^H f_1$ λ -a.s. Then*

(1) *$Y_n(t)$ converges a.s. (to $Y(t)$),*
(2) *$Y(t) \stackrel{\mathcal{D}}{=} \int f_t d\Lambda$,*
(3) *$\{Y(t); t \in T\}$ is H-ss stochastic process.*

Proof. Let $f \in L_{\phi_p}, p \geq 1$. First note that $C_f(a)$ is well-defined. Indeed,

$$\begin{aligned}
&\int_0^a \int_S R^{(1)}(r, s) f(s) \lambda^{(1)}(ds) dr \\
&\leq a + \int_0^a \int_S |R^{(1)}(r, s) f(s)| I_{(|R^{(1)}(r,s) f(s)| > 1)} \lambda^{(1)}(ds) dr \\
&= a + \int_{\{|x| > 1\}} (|x|)^p F_f(dx) \\
&= a + \int_S \int_{\{|f(s)x| > 1\}} (|f(s)x|)^p Q(s, dx) \lambda(ds) \\
&= a + \int_S \Phi_p(|f(s)|, s) \lambda(ds) \\
&< \infty.
\end{aligned}$$

We showed in the proof of Theorem 2.4 that F_f is a Lévy measure. Put

$$H(r,s) = R^{(1)}(r,s)f(s) \text{ in [6, Theorem 3.1]}$$

Then we get (1) and

$$\begin{aligned}
&\mathcal{L}(\sum_{j=1}^{\infty} R^{(1)}(\Gamma_j,\xi_j)f(\xi_j) - \lim_{n\to\infty} C_f(\Gamma_n)) \\
&= \mathcal{L}(\int f d\Lambda) \\
&= \exp\{\int \{e^{iuR^{(1)}(r,s)f(s)} - 1 - iuR^{(1)}(r,s)f(s)\}\lambda^{(1)}(ds)dr\}.
\end{aligned}$$

Let $f = \sum_{i=1}^{m} a_i f_{t_i}$. Then we get (2), since we have

$$\begin{aligned}
\sum_{i=1}^{m} a_i \int f_{t_i} d\Lambda &= \int f d\Lambda \\
&= \sum_{j=1}^{\infty} R^{(1)}(\Gamma_j,\xi_j)f(\xi_j) - \lim_{n\to\infty} C_f(\Gamma_n) \\
&= \sum_{j=1}^{\infty} R^{(1)}(\Gamma_j,\xi_j)\{\sum_{i=1}^{m} a_i f_{t_i}(\xi_j)\} - \lim_{n\to\infty} C_f(\Gamma_n) \\
&= \sum_{i=1}^{m} a_i\{\sum_{j=1}^{\infty} R^{(1)}(\Gamma_j,\xi_j)f_{t_i}(\xi_j) - \lim_{n\to\infty} C_{f_{t_i}}(\Gamma_n)\} \\
&= \sum_{i=1}^{m} a_i Y(t_i).
\end{aligned}$$

By the same argument as in Theorem 3.2,

$$\{Y(t); t\in T\} \stackrel{\mathcal{D}}{=} \{\int f_t d\Lambda; t \in t\}$$

is H-ss stochastic integral process.

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