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GEOMETRIC CHARACTERIZATIONS OF CONCENTRATION POINTS FOR MÖBIUS GROUPS

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1. Introduction

Although the study of the limit points of discrete groups of Möbius transformations has been a fertile area for many decades, there are some very natural topological properties of the limit points which appear not to have been previously examined. Let Γ be a nonelementary discrete group of hyperbolic isometries acting on the Poincaré disc B^m, $m \geq 2$, and let $p \in \partial B^m$ be a limit point of Γ. By a neighborhood of p, we will always mean an open neighborhood of p in ∂B^m.

DEFINITION 1.1. One says that a neighborhood U of p *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U)$ and $\gamma(U) \subseteq V$. If in addition the element γ can always be selected so that $p \in \gamma(V)$, then one says that U can be concentrated *with control.*
Equivalently, U can be concentrated at p if and only if the set of translates of U contains a local basis for the topology of ∂B^m at p. It is not difficult to show (see [**M1**]) that every limit point has a disconnected neighborhood that can be concentrated. So the weakest reasonable concept of concentration is the following.

DEFINITION 1.2. The limit point p is called a *weak concentration point* for Γ if there exists a connected open set U that can be concentrated at p.

Weak concentration points are studied in [**M1**]. It turns out that for a geometrically finite group, every limit point is a weak concentration

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point, while for any group, all but countably many limit points are weak concentration points. A little more restrictive condition is that every sufficiently small connected neighborhood can be concentrated:

DEFINITION 1.3. The limit point p is called a *concentration point* for Γ if there exists a neighborhood W of p such that every neighborhood U of p with $U \subseteq W$ can be concentrated at p.

Every concentration point for a Fuchsian group is a conical limit point (see Remark 4.3 in [**M2**]). Where the same is true in higher dimensions is an open question.

Concentration with control is yet more restrictive. It follows from the definition that (1) there exists a neighborhood of p which can be concentrated with control if and only if there is a connected which can be concentrated with control (take the connected component U that contains p, and require that $\gamma^{-1}(p) \in U \cap V$), and (2) if a neighborhood of p can be concentrated with control, then every smaller neighborhood can be concentrated with control. As a consequence of these observations, there is only one concept involving concentrated with control:

DEFINITION 1.4. The limit point p is called a *controlled concentration point* for Γ if there exists a neighborhood U of p which can be concentrated with control at p.

Concentration with control and related topics are studied in [**AHM**], [**H1**] and [**H2**]. Three characterizations of controlled concentration points are given in [**AHM**]. Each characterization shows easily that in all dimension, a controlled concentration point must be a conical limit point. However, examples are given there of conical limit points of 2-generator Schottky groups which are not controlled concentration points.

It turns out that a slight weakening of the definition of concentration point is of importance in dimension 2. A limit point of a Fuchsian group is called a *geodesic separation point* if there is a connected neighborhood W of p in $\overline{B^2}$. Equivalently, if U is any sufficiently small connected neighborhood of p, then for every neighborhood V of p there exists $\gamma \in \Gamma$ so that either $p \in \gamma(U) \subseteq V$, or $p \in \gamma(\partial B^2 - \overline{U}) \subseteq V$.

Clearly a concentration point must be a geodesic separation point and every geodesic separation point is a conical limit point (see Proposition 4.2 in [**M2**]).

We will assume familiarity with the basic concepts of Möbius groups as exposed, for example, in [**B**].

2. Controlled concentration points

Now we give one of geometric characterization of controlled concentration points analytic proof of which can be found in [**B-H-M**]. Note that if there exists a sequence of distinct elements $\{\gamma_n\}$ of Γ such that $\{\gamma_n(0)\}$ converge to $r \in \partial B^m$ then so is for all $x \in B^m$.

THEOREM 2.1. *A limit point p is a controlled concentration point for Γ if and only if there are a sequence $\{\gamma_n\}$ of elements of Γ and a point $r \in \partial B^m$ with $r \neq p$ so that $\{\gamma_n(p)\}$ converges to p and $\{\gamma_n(0)\}$ converges (in $B^m \cup \partial B^m$) to r.*

Proof. Suppose first that p is a controlled concentration point. Fix a neighborhood U that can be concentrated with control at p. Since any smaller neighborhood can be concentrated with control, we may assume that U is a round ball in ∂B^m,whose boundary is spanned by an $(m-1)$-dimensional hyperbolic hyperplane H in B^m, and that the closure of H separates U from 0 in $B^m \cup \partial B^m$. Let V_n be a sequence of neighborhoods of p whose diameters limit to 0, and let γ_n be elements of Γ such that $p \in \gamma_n(V_n)$ and $\gamma_n(U) \subseteq V_n$. Then $\gamma_n^{-1}(p) \in V_n$ so the $\gamma_n^{-1}(p)$ converge to p, while for all sufficiently large n, $\gamma_n^{-1}(0)$ converge to a point $r \in \partial B^m$ with $r \neq p$. Conversely, suppose that the point r and the sequence $\{\gamma_n\}$ exist. Let λ be the geodesic in B^m running from r to p, and let v_0 be a unit (for the hyperbolic metric) tangent vector to λ at the point x_0 (with v_0 oriented in the direction of p). Let U be the neighborhood of p bounded by the boundary of the $(m-1)$-dimensional geodesic hyperplane H_0 through x_0 with normal vector v_0. We will show that U can be concentrated with control at p. Let V be any neighborhood of p. We may replace V by any smaller neighborhood, so we may assume that V is bounded by the boundary of an $(m-1)$-dimensional geodesic hyperplane H through a point x_1 on λ and perpendicular to λ there. Let d be the hyperbolic distance from x_0 to x_1. Fix a metric on the unit tangent space to the Poincaré disc, invariant under all hyperbolic isometries, and fix a small ϵ. There is a positive constant k so that if α and β are geodesic rays with initial unit tangent vectors within distance d, then the hyperplanes perpendicular to α at its initial point and at the point at distance k from the initial point must both intersect the geodesic containing β, making angles within ϵ of $\pi/2$ with it at the intersection points. For any sufficiently large n, there exists a tangent vector w_0 to $\gamma_n(\lambda)$ which lies within distance k of v_0. Then, the vector $\gamma_n^{-1}(v_0)$ lies within distance k of the vector $\gamma_n^{-1}(w_0)$ on λ. Because of the property of k, $\gamma_n^{-1}(H_0)$ and $\gamma_n^{-1}(H_0)$ separates p from H, forcing $\gamma_n^{-1}(U) \subseteq V$. This completes the proof.

THEOREM 2.2. *Let Γ be a torsionfree discrete group of Möbius transformations acting on the Poincaré disc B^m, let $\pi : B^m \to B^m/\Gamma$ be the quotient map. Let $y_0 \in B^m$, and let p be a point in ∂B^m. Let $\alpha : [0, \infty) \to B^m$ be the geodesic ray from y_0 to p, parameterized at unit speed. Suppose further that there exist numbers t_i, with $\alpha(t_i)$ limiting to p, so that in the tangent bundle $T(B^m/\Gamma)$, the images $d\pi(\alpha'(t_i))$ converge to $d\pi(\alpha'(0))$. Then p is a controlled concentration point for Γ.*

Proof. By Theorem 2.1, p is a controlled concentration point for Γ if and only if there exist a sequence γ_i of elements of Γ and a point $r \in \partial B^m$, with $r \neq p$, such that the $\gamma_i(p)$ converge to p and the $\gamma_i(y_0)$ converge to r. Suppose the hypothesis of Theorem 2.2 holds. Let r be the endpoint other than p of the geodesic that contains α. For large i, there are elements γ_i of Γ that translate $\alpha'(t_i)$ very close to $\alpha'(0)$ in TB^m. Then these satisfy the condition of Theorem 2.1. This completes the non-self-contained proof of Theorem 2.2.

To prove Theorem 2.2 directly from the definition of controlled concentration point, consider the hyperbolic codimension 1 hyperplane through y_0 perpendicular to α, and let U be the neighborhood of p in ∂B^m which is one of the components of the complement of the boundary of the hyperplane. For fixed positive n, let V_n be the smaller neighborhood of p whose boundary is the boundary of the hyperplane perpendicular to α and crossing it at hyperbolic distance n from y_0. The hypothesis of Theorem 2.2 shows there are elements $\gamma_i \in \Gamma$ which translate $\alpha'(0)$ close to $\alpha'(t_i)$. For all sufficiently large i, the geodesic $\gamma(\alpha)$ follows along very close to α for more than distance n-close enough so that $p \in \gamma_i(V_n)$. By making i even larger if it is necessary, one can ensure that $\gamma_i(U) \subseteq V_n$, showing that p is a controlled concentration point. This completes the self-contained proof.

THEOREM 2.3. *If p is a controlled concentration point for Γ and Γ_0 is a subgroup of finite index in Γ, then p is also a controlled concentration point for Γ_0.*

Proof. Assume that p is a controlled concentration point for Γ, then by Theorem 2.1 there are a sequence $\{\gamma_n\} \subseteq \Gamma$ and a point $r \in \partial B^m$ $(r \neq p)$ such that $\{\gamma_n(p)\}$ converge to p and $\{\gamma_n(0)\}$ converge to r. Since Γ_0 is a subgroup of finite index, Γ can be written as the union of right coset $\Gamma_0 g_i$ for $i \in \{1, 2, \ldots n\}$. But γ_n is an infinite sequence, hence one of $\Gamma_0 g_i$, say $\Gamma_0 g_k$ contains infinitely many elements of γ_n. Therefore by passing to a subsequence γ_{n_i} of γ_n, $\gamma_{n_i} \in \Gamma_0 g_k$. From the hypothesis and from the remark prior to the geometric characterization, we have a sequence $\{\gamma_{n_i} g_k^{-1}\}$ such that $\{\gamma_{n_i} g_k^{-1}(p)\}$ converge to p and $\{\gamma_{n_i} g_k^{-1}(0)\}$ converge to r. This completes the proof of Theorem 2.3.

Question: It is well known that if N is a nontrivial normal subgroup of Γ then the limit set of Γ is the same as that of N. Hence the natural question is whether it is true that the set of controlled concentration points of Γ is equal to that of N.

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