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A NOTE ON SPECTRAL CHARACTERIZATIONS OF COSYMPLECTIC FOLIATIONS

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1. Introduction

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ ([9]). Aside from the Laplacian $\triangle_g$ associated to the metric g, there is another differential operator, the Jacobi operator J_D, which is a second order elliptic operator acting on sections of the normal bundle. Its spectrum is discrete as a consequence of the compactness of M. The study of the spectrum of $\triangle_g$ acting on functions or forms has attracted a lot of attention. In this point of view, the present authors [7] have studied the spectrum of the Laplacian and the curvature of a compact orientable cosymplectic manifold. On the other hand, S. Nishikawa, Ph. Tondeur and L. Vanhecke [6] studied the spectral geometry for Riemannian foliations. The purpose of the present paper is to study the relation between two spectra and the transversal geometry of cosymplectic foliations.

We shall be in C^∞-category. Manifolds are assumed to be connected.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be as above. Let ∇ be the Levi-Civita connection with respect to g_M. Then the tangent bundle TM over M has an integrable subbundle E which is given by $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ is defined by $Q = TM/E$. We have a splitting σ of the exact sequence

$$0 \longrightarrow E \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0,$$

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where $\sigma(Q)$ is orthogonal complement bundle $E^{\perp}$ of E in TM ([4]). Then g_M induces a metric g_Q on Q :

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)) \tag{2.1}$$

for any $s, t \in \Gamma(Q)$, where $\Gamma(*)$ denotes the set of all sections of $*$.

In a flat chart $U(x^i, x^a)$ with respect to $\mathcal{F}$ ([9]), a local frame $\{X_i, X_a\} = \{\partial/\partial x^i, \partial/\partial x^a - A^j{}_a \partial/\partial x^j\}$ is called the *basic adapted frame* to $\mathcal{F}$ ([9,10,11]). Here $A^j{}_a$ are functions on U with $g_M(X_i, X_a) = 0$. It is clear that $\{X_i\}$ (resp. $\{X_a\}$) spans $\Gamma(E|_U)$ (resp. $\Gamma(E^{\perp}|_U)$). From now on, we omit "$|_U$" for simplicity.

We set

$$\begin{aligned} &g_{ij} = g_M(X_i, X_j), \ g_{ab} = g_M(X_a, X_b), \\ &(g^{ij}) = (g_{ij})^{-1}, \ (g^{ab}) = (g_{ab})^{-1}. \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that $g_Q(\pi(X_a), \pi(X_b)) = g_{ab}$.

A connection D in Q is defined by

$$D_X s = \begin{cases} \pi([X,Y]), & X \in \Gamma(E), \quad s \in \Gamma(Q) \text{ with } \pi(Y) = s, \\ \pi(\nabla_X Y_s), & X \in \Gamma(E^{\perp}), \quad s \in \Gamma(Q) \text{ with } Y_s = \sigma(s) \end{cases} \tag{2.3}$$

([4]). The connection D in Q is torsion free and metrical with respect to g_Q. The transversal curvature R_D of D is defined by

$$R_D(X,Y)s = D_X D_Y s - D_Y D_X s - D_{[X,Y]} s \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$ and $s \in \Gamma(Q)$. Since $i(X)R_D = 0$ for any $X \in \Gamma(E)$ ([4]), we can define the transversal Ricci operator $\rho_D : \Gamma(Q) \to \Gamma(Q)$ and the transversal scalar curvature σ_D of D by

$$\rho_D(s) = g^{ab} R_D(\sigma(s), X_a)\pi(X_b), \quad s \in \Gamma(Q) \tag{2.5}$$

and

$$\sigma_D = g^{ab} g_Q(\rho_D(\pi(X_a), \pi(X_b))), \tag{2.6}$$

respectively ([5]).

For a distinguished chart $U \subset M$ the leaves of $\mathcal{F}$ in U are given as the fibers of a Riemannian submersion $f : U \to V \subset N$ onto an open subset V of a model Riemannian manifold N. Since $\dim E = p$, $\dim Q = q$ and $\dim M = p + q = m$, $\dim N = q$. For overlapping charts $U_\alpha \cap U_\beta$ the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ {f_\beta}^{-1}$ on N are isometries. The transversal geometry is heuristically the geometry of the model space N. Technically the corresponding transversal curvature data are R_D, ρ_D and σ_D. $\mathcal{F}$ is said to be *(transversally) Einstein* if the model space N is Einstein, that is $\rho_D = \frac{1}{q}\sigma_D \circ id$ with constant transversal scalar curvature σ_D, where $q = \operatorname{codim}\mathcal{F}$. Similarly $\mathcal{F}$ is said to be *of constant (transversal) ϕ-holomorphic sectional curvature* if the model space N is of constant ϕ-holomorphic sectional curvature. In the sequel we will mostly delete the adjective "transversal" in similar situations. Furthermore, $\mathcal{F}$ is called a *cosymplectic foliation* if it is modeled on a cosymplectic manifold.

For a Riemannian foliation $\mathcal{F}$ with metric g_Q and canonical connection D on Q the usual calculus for Q-valued forms on M applies. Let $\Delta = d_D^* d_D$ be the Laplacian acting on sections of $\Gamma(Q)$. Then the *Jacobi operator* of a Riemannian foliation $\mathcal{F}$ is given by $J_D v = (\Delta - \rho_D)v$ for $v \in \Gamma(Q)$ ([5]). With respect to the natural scalar product on $\Gamma(Q)$ it is strongly elliptic of the second order with leading symbol g. It has a discrete spectrum with finite multiplicities.

Consider the case of a transversally oriented codimension $q = 1$ foliation $\mathcal{F}$. Then the Ricci operator ρ_D vanishes. Sections of Q can be identified with functions on M, and it is easy to see that then an eigenfunction of the Jacobi operator on $\Gamma(Q)$ corresponds to an eigenfunction of the ordinary Laplacian on M associated to the same eigenvalue. Thus no new information is encoded in $\operatorname{Spec}(\mathcal{F}, J_D)$. Throughout the rest of the paper we assume therefore $q \geq 2$.

Consider the semigroup $e^{-t\Delta_g}$, and the semigroup e^{-tJ_D} given by

$$e^{-tJ_D}u(x) = \int_M K(t,x,y,J_D)\, u(y)\, d\, Vol(y),$$

where $K(t,x,y,J_D) \in \operatorname{Hom}(Q_y, Q_x)$ is the kernel function. We have asymptotic expansions for the corresponding L^2-trace of $e^{-t\Delta_g}$ and the

L^2-trace

$$Tr\ e^{-tJ_D} = \int_M tr_{Q_x}\ K(t,x,x,J_D)\ d\ Vol(x)$$

for $t \downarrow 0$:

$$(2.7)\qquad \begin{aligned} Tr\ e^{-t\Delta_g} &= \sum_{i=1}^{\infty} e^{-t\lambda_i}\ \widetilde{t \downarrow 0}\ (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(\Delta_g), \\ Tr\ e^{-tJ_D} &= \sum_{i=1}^{\infty} e^{-t\mu_i}\ \widetilde{t \downarrow 0}\ (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n b_n(J_D), \end{aligned}$$

where

$$a_n(\Delta_g) = \int_M a_n(x,\Delta_g)\ d\ Vol(x),$$

$$b_n(J_D) = \int_M b_n(x,J_D)\ d\ Vol(x)$$

are invariants of Δ_g and J_D depending only on the discrete spectra

$$\mathrm{Spec}(M,g) = \{0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots \uparrow \infty\},$$

$$\mathrm{Spec}(\mathcal{F},J_D) = \{\mu_1 \le \mu_2 \le \cdots \le \mu_i \le \cdots \uparrow \infty\}.$$

We state the classical formulas for $a_n(\Delta_g)$ ([8]) (see also [1, 3]). Using the local formulas for $b_n(x,J_D)$ given by Gilkey ([3, p. 327]) we also obtain the $b_n(J_D)$. The curvature data associated to (M,g) are denoted by R_M, ρ_M, σ_M and in this paper we take the convention

$$R_M(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for X, Y tangent vector fields on M. We have

LEMMA 2.1([6]). *Let $\mathcal{F}$ be a smooth Riemannian foliation of codimension $q \ge 2$ on (M,g) with volume form $\mu = d\ Vol$. Then*

$$a_0(\Delta_g) = VolM,$$

$$a_1(\Delta_g) = \frac{1}{6}\int_M \sigma_M \mu,$$

$$a_2(\Delta_g) = \frac{1}{360}\int_M [2|R_M|^2 - 2|\rho_M|^2 + 5{\sigma_M}^2]\mu$$

and

$$b_0(J_D) = q\ VolM,$$
$$b_1(J_D) = q\ a_1(\Delta_g) + \int_M \sigma_D \mu,$$
$$b_2(J_D) = q\ a_2(\Delta_g) + \frac{1}{12}\int_M [2\sigma_M\sigma_D + 6|\rho_D|^2 - |R_D|^2]\mu.$$

Note that $\int_M \sigma_M \mu$ is the *total scalar curvature* of (M, g) and we call $\int_M \sigma_D \mu$ the *total scalar curvature* of the foliation.

DEFINITION ([6]). The Riemannian foliations $(M, g, \mathcal{F})$ and $(M_0, g_0, \mathcal{F}_0)$ are said to be isospectral if

$$\mathrm{Spec}(M, g) = \mathrm{Spec}(M_0, g_0),$$
$$\mathrm{Spec}(\mathcal{F}, J_D) = \mathrm{Spec}(\mathcal{F}_0, J_{D_0}).$$

From (2.7) and Lemma 2.1 we get the following results.

LEMMA 2.2([6]). *Let $(M, g, \mathcal{F})$ and $(M_0, g_0, \mathcal{F}_0)$ be isospectral Riemannian foliations. Then we have*

1) $\dim M = \dim M_0$;
2) *$Vol\,M = Vol\,M_0$;*
3) *(M, g) and (M_0, g_0) have equal total scalar curvature ;*
4) codim$\mathcal{F}$=codim$\mathcal{F}_0$, *and hence $\mathcal{F}$ and $\mathcal{F}_0$ have the same energy ;*
5) *$\mathcal{F}$ and $\mathcal{F}_0$ have equal total scalar curvature ;*
6) $\int_M [2|R_M|^2 - 2|\rho_M|^2 + 5{\sigma_M}^2]\mu = \int_{M_0} [2|R_{M_0}|^2 - 2|\rho_{M_0}|^2 + 5{\sigma_{M_0}}^2]\mu_0$;
7) $\int_M [2\sigma_M\sigma_D + 6|\rho_D|^2 - |R_D|^2]\mu = \int_{M_0} [2\sigma_{M_0}\sigma_{D_0} + 6|\rho_{D_0}|^2 - |R_{D_0}|^2]\mu_0$.

For 4) note that the energy $E(\mathcal{F}) = \frac{1}{2}q\ \mathrm{Vol}\,M$ ([4, p.116]).

3. Transversal geometry and cosymplectic foliations

Let $(M, g, \mathcal{F})$ be a cosymplectic foliation of codimension $2q + 1 \geq 5$. A tensor field H_D on Q is defined by

$$\begin{aligned} H_D(x,y)z =& R_D(x,y)z - \frac{\sigma_D}{4q(q+1)}\{g_Q(y,z)x - g_Q(x,z)y \\ &+ g_Q(\phi(y),z)\phi(x) - g_Q(\phi(x),z)\phi(y) - 2g_Q(\phi(x),y)\phi(z) \\ &- \eta_D(y)\eta_D(z)x + g_Q(x,z)\eta_D(y)\xi_D \\ &- g_Q(y,z)\eta_D(x)\xi_D + \eta_D(x)\eta_D(z)y\}, \end{aligned}$$

where $x, y, z \in \Gamma(Q)$. Then we have

$$|H_D|^2 = |R_D|^2 - \frac{2}{q(q+1)}{\sigma_D}^2. \tag{3.1}$$

A cosymplectic foliation is of constant ϕ-holomorphic sectional curvature if and only if $H_D = 0$, provided $q \geq 2$.

A tensor field E_D on Q is defined by

$$E_D(x) = \{\rho_D - \frac{\sigma_D}{2q}(I - \eta_D \otimes \xi_D)\}(x),$$

where $x \in \Gamma(Q)$ and I denotes the identity transformation. Then we have

$$|E_D|^2 = |\rho_D|^2 - \frac{1}{2q}{\sigma_D}^2. \tag{3.2}$$

A cosymplectic foliation is said to be η_D-*Einstein foliation* if $E_D = 0$. For any η_D-Einstein cosymplectic foliation, σ_D is constant, provided $q \geq 2$.

We also consider the so-called *cosymplectic Bochner curvature tensor field* $\bar{B}_D$ associated to R_D on Q by(cf.[2])

$$\begin{aligned}
\bar{B}_D(x,y)z = R_D(x,y)z &- \frac{1}{2(q+2)}\{g_Q(\rho_D(y),z)x - g_Q(\rho_D(x),z)y \\
&+ g_Q(y,z)\rho_D(x) - g_Q(x,z)\rho_D(y) + g_Q(S_D(y),z)\phi(x) \\
&- g_Q(S_D(x),z)\phi(y) + g_Q(\phi(y),z)S_D(x) - g_Q(\phi(x),z)S_D(y) \\
&- 2g_Q(\phi(x),y)S_D(z) - 2g_Q(S_D(x),y)\phi(z) \\
&- g_Q(\rho_D(y),z)\eta_D(x)\xi_D + g_Q(\rho_D(x),z)\eta_D(y)\xi_D \\
&- \eta_D(y)\eta_D(z)\rho_D(x) + \eta_D(x)\eta_D(z)\rho_D(y)\} \\
&+ \frac{\sigma_D}{4(q+1)(q+2)}\{g_Q(y,z)x - g_Q(x,z)y - \eta_D(y)\eta_D(z)x \\
&+ \eta_D(x)\eta_D(z)y - g_Q(y,z)\eta_D(x)\xi_D + g_Q(x,z)\eta_D(y)\xi_D \\
&+ g_Q(\phi(y),z)\phi(x) - g_Q(\phi(x),z)\phi(y) - 2g_Q(\phi(x),y)\phi(z)\},
\end{aligned}$$

where $S_D := \rho_D \circ \phi$ and $x, y, z \in \Gamma(Q)$.

A cosymplectic foliation with $\bar{B}_D = 0$ is said to be *cosymplectic Bochner flat.* We can easily see that

$$|\bar{B}_D|^2 = |R_D|^2 - \frac{8}{q+2}|\rho_D|^2 + \frac{2}{(q+1)(q+2)}{\sigma_D}^2 \tag{3.3}$$

and

$$|\bar{B}_D|^2 = |H_D|^2 - \frac{8}{q+2}|E_D|^2. \tag{3.4}$$

Thus from (3.4) we have the following

THEOREM 3.1. *Let $(M, g, \mathcal{F})$ be a cosymplectic foliation of codimension$\geq$5. Then $(M, g, \mathcal{F})$ is of constant ϕ-holomorphic sectional curvature if and only if it is η_D-Einstein and cosymplectic Bochner flat.*

THEOREM 3.2. *Let $(M, g, \mathcal{F})$ and $(M_0, g_0, \mathcal{F}_0)$ be isospectral cosymplectic η_D-Einstein foliations of codimension $\geq$5. Then $(M, g, \mathcal{F})$ is of constant ϕ-holomorphic sectional curvature c if and only if $(M_0, g_0, \mathcal{F}_0)$ is of constant ϕ-holomorphic sectional curvature c.*

Proof. Since $E_D = E_{D_0} = 0$, it follows from (3.2) that

$$|\rho_D|^2 = \frac{1}{2q}{\sigma_D}^2, \qquad |\rho_{D_0}|^2 = \frac{1}{2q_0}{\sigma_{D_0}}^2, \tag{3.5}$$

where σ_D and σ_{D_0} are constant. Using Lemma 2.2 we obtain $q = q_0$ and $\sigma_D = \sigma_{D_0}$. Thus from (3.5) we have

$$|\rho_D|^2 = |\rho_{D_0}|^2. \tag{3.6}$$

Since the total scalar curvatures are also equal, the equation 7) in Lemma 2.2 implies

$$\int_M |R_D|^2\mu = \int_{M_0} |R_{D_0}|^2\mu_0$$

with the help of (3.6), and consequently

$$\int_M [|R_D|^2 - \frac{2}{q(q+1)}{\sigma_D}^2]\mu = \int_{M_0} [|R_{D_0}|^2 - \frac{2}{q(q+1)}{\sigma_{D_0}}^2]\mu_0. \tag{3.7}$$

Taking account of (3.1), (3.7) reduces

$$\int_M |H_D|^2\mu = \int_{M_0} |H_{D_0}|^2\mu_0,$$

which implies Theorem.

THEOREM 3.3. *Let $(M, g, \mathcal{F})$ and $(M_0, g_0, \mathcal{F}_0)$ be isospectral cosymplectic Bochner flat foliations of codimension ≥ 5 with constant scalar curvature σ_D. Then $(M, g, \mathcal{F})$ has constant ϕ-holomorphic sectional curvature c if and only if $(M_0, g_0, \mathcal{F}_0)$ has constant ϕ-holomorphic sectional curvature c.*

Proof. From Lemma 2.2 we easily check that

$$\int_M [|R_D|^2 - 6|\rho_D|^2]\mu = \int_{M_0} [|R_{D_0}|^2 - 6|\rho_{D_0}|^2]\mu_0,$$

which implies

$$\begin{aligned}(3.8)\qquad &\int_M [|\bar{B}_D|^2 - \frac{2(3q+2)}{q+2}|E_D|^2 - \frac{3q+1}{q(q+1)}{\sigma_D}^2]\mu \\ &= \int_{M_0} [|\bar{B}_{D_0}|^2 - \frac{2(3q+2)}{q+2}|E_{D_0}|^2 - \frac{3q+1}{q(q+1)}{\sigma_{D_0}}^2]\mu_0\end{aligned}$$

by virtue of (3.2) and (3.3). Since $\bar{B}_D = \bar{B}_{D_0} = 0$, (3.8) may be written as

$$\begin{aligned}(3.9)\qquad &\frac{2(3q+2)}{q+2}\left(\int_M |E_D|^2\mu - \int_{M_0} |E_{D_0}|^2\mu_0\right) \\ &= \frac{3q+1}{q(q+1)}\left(\int_{M_0} {\sigma_{D_0}}^2\mu_0 - \int_M {\sigma_D}^2\mu\right).\end{aligned}$$

From now on assume that $(M_0, g_0, \mathcal{F}_0)$ is of constant ϕ-holomorphic sectional curvature c. Then (3.9) yields

$$(3.10)\qquad \frac{2(3q+2)}{q+2}\int_M |E_D|^2\mu = \frac{3q+1}{q(q+1)}\left(\int_{M_0} {\sigma_{D_0}}^2\mu_0 - \int_M {\sigma_D}^2\mu\right),$$

from which, together with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(3.11)\qquad VolM \int_M {\sigma_D}^2\mu &\geq \left(\int_M \sigma_D\mu\right)^2 = \left(\int_{M_0} \sigma_{D_0}\mu_0\right)^2 \\ &= VolM \int_{M_0} {\sigma_{D_0}}^2\mu_0.\end{aligned}$$

Thus it follows from (3.10) and (3.11) that

$$\int_M |E_D|^2 \mu \leq 0,$$

which implies $E_D = 0$. So, $(M, g, \mathcal{F})$ is η_D-Einstein foliation and consequently is of constant ϕ-holomorphic sectional curvature c.

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