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AN HEAT EQUATION APPROACH TO DISTRIBUTIONS WITH L^p GROWTH

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1. Introduction

In this paper we characterize the distributions with L^p-growth. Moreover, we give a much simpler structure theorem than already known so far.

The space $\mathcal{D}'_{L^p}$ of distributions with L^p growth has been studied by several authors in the past few years. Pahk [P] gave necessary and sufficient conditions for a convolution operator in $\mathcal{D}'_{L^p}$ to be hypoelliptic. Ortnar and Wagner [OW] considered the convolution of elements in $\mathcal{D}'_{L^p}$. Besides, Abdullah and Pilipovic [AP] characterized the bounded subset in this space.

In this work, we characterize the element u of $\mathcal{D}'_{L^p}, 1 < p < +\infty$, by the solution of the heat equation. In fact, it will be shown that

$$\mathcal{D}'_{L^p} \cong \mathcal{H}_{L^p}(\mathbb{R}^n_+)$$

where $\mathcal{H}_{L^p}(\mathbb{R}^{n+1}_+)$ is the space of the solutions $U(x,t)$ of the heat equation in $\mathbb{R}^{n+1}_+ = \{(x,t)|x \in \mathbb{R}^n, t > 0\}$ with

$$\|U(x,t)\|_{L^p(\mathbb{R}^n_x)} \le Ct^{-N}$$

for some constants $C > 0$ and $N > 0$. In addition to this result we show that every $u \in \mathcal{D}'_{L^p}$ can be written as

$$u = \Delta^m g(x) + h(x)$$

where $g(x)$ and $h(x)$ are bounded continuous functions on $\mathbb{R}^n$ belonging to L^p and Δ is the Laplacian. This, actually, improves the already known structure theorem of $\mathcal{D}'_{L^p}$ (see (2.1)).

Throughout this paper we use general notations such as $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha \in \mathbb{N}_0^n$ where $\mathbb{N}_0$ is the set of all nonnegative integers.

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2. Main Results

Following Schwartz [S] we introduce briefly the space of distributions with L^p-growth.

DEFINITION 2.1. (i) Let p be a real member such that $1 \leq p \leq +\infty$. We denote by $\mathcal{D}_{L^p}(\mathbb{R}^n)$ the space of all functions $\phi \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha \phi \in L^p(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ equipped with the coarsest locally convex topology for which the maps

$$\partial^\alpha : \mathcal{D}_{L^p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

are continuous for all $\alpha \in \mathbb{N}_0^n$. The topology of $\mathcal{D}_{L^p}(\mathbb{R}^n)$ coincides trivially with the one defined by the countable family of norms

$$\|\phi\|_{m,p} = \left[\sum_{|\alpha| \leq m} \|\partial^\alpha \phi\|_{L^p}^p \right]^{1/p}, \quad m \in \mathbb{N}.$$

In fact, it is obvious that $\mathcal{D}'_{L^p}, 1 \leq p \leq +\infty$, the dual of $\mathcal{D}_{L^q}$, where $1/p + 1/q = 1$.

It is well known that $\mathcal{D}'_{L^p}$ is a subspace of the space $\mathcal{D}'$ of the Schwartz distributions and every $T \in \mathcal{D}'_{L^p}$ can expressed by the sum

$$T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha \tag{2.1}$$

for some $m \in \mathbb{N}$ where f_α are bounded continuous functions belonging to L^p.

Now we denotes by $E(x,t)$ the n-dimensional heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Then using the Cauchy integral formula we can obtain the following estimate:

(2.2)

$$|\partial_x^\alpha E(x,t)| \leq C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-|x|^2/8t), \quad x \in \mathbb{R}^n, \quad t > 0.$$

for some constant $C > 0$ (see [M, p.622] for the details). It then follows from the estimate that $E(\cdot, t)$ belongs to $\mathcal{D}_{L^p}(\mathbb{R}^n)$ for $1 \leq p \leq +\infty$ and for each $t > 0$.

LEMMA 2.2. *For every $\phi \in \mathcal{D}_{L^p}, 1 \le p \le +\infty$ let*

$$\phi_t(x) = \int E(x-y,t)\phi(y)dy, \quad t > 0.$$

Then $\phi_t \to \phi$ in $\mathcal{D}_{L^p}$ as $t \to 0+$.

Proof. We note that $E(x,t)$ plays a role of approximation unit in L^p as $t \to 0+$. In other words, if $f(x) \in L^p$ then it is true that $f * E(x,t) \in L^p$ and $f * E(x,t) \to f$ in L^p as $t \to 0+$. From these facts we obtain

$$\partial^\alpha \phi_t(x) = (\partial^\alpha \phi) * E(x,t) \in L^p$$

and

$$\partial^\alpha \phi_t(x) \longrightarrow \partial^\alpha \phi \quad \text{in} \quad L^p \quad \text{as} \quad t \to 0+,$$

which means that $\phi_t(x) \in \mathcal{D}_{L^p}$ and $\phi_t(x) \to \phi(x)$ in $\mathcal{D}_{L^p}$ as $t \to 0+$.

Let $u \in \mathcal{D}'_{L^p}$. Then the function

$$U(x,t) = u_y(E(x-y,t)), \quad x \in \mathbb{R}^n, \quad t > 0,$$

is a well defined C^∞ function since $E(x-\cdot,t)$ belongs to $\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$, for every $(x,t) \in \mathbb{R}^{n+1}_+ = \{(x,t)|x \in \mathbb{R}^n, t > 0\}$.

THEOREM 2.3. *Let $u \in \mathcal{D}'_{L^p}, 1 < p < +\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $U(x,t) = u_y(E(x-y,t))$ belongs to $C^\infty(\mathbb{R}^{n+1})$ and satisfies the followings:*

(i) $(\partial_t - \Delta)U(x,t) = 0$ *in* $\mathbb{R}^{n+1}_+$,

(ii) *There exist constants $C > 0$ and $N > 0$ such that*

$$\|U(x,t)\|_{L^p(\mathbb{R}^n_x)} \le Ct^{-N} \quad in \quad \mathbb{R}^{n+1}_+ \tag{2.3}$$

(iii) *$U(x,t) \to u$ as $t \to 0+$ in the sense that*

$$u(\phi) = \lim_{t\to 0+} \int_{\mathbb{R}^n} U(x,t)\phi(x)dx, \quad \phi \in \mathcal{D}_{L^q}. \tag{2.4}$$

Conversely, every C^∞ function $U(x,t)$ defined in $\mathbb{R}^{n+1}_+$ satisfying condition (i) and (ii) can be written

$$U(x,t) = u_y(E(x-y,t))$$

in $\mathbb{R}^{n+1}_+$ with a unique element $u \in \mathcal{D}'_{L^p}$.

Proof. Let $u \in \mathcal{D}'_{L^p}$. Then if follows from Lemma 2.2 that we obtain, for every $\phi \in \mathcal{D}_{L^q}$,

$$\int_{\mathbb{R}^n} U(x,t)\phi(x)dx = u_y(\phi_t(x))$$

which gives (2.4).

Since $u \in \mathcal{D}'_{L^p}$, in view of (2.1) we may write u as

$$u = \sum_{|\alpha|\le m} \partial^\alpha f_\alpha$$

where f_α are bounded continuous function belonging to L^p. Then by an elementary calculation on the convolution we obtain

$$\begin{aligned} \|U(x,t)\|_{L^p} &= \|u_y(E(x-y,t))\|_{L^p} \\ &\le \sum_{|\alpha|\le m} \|\partial^\alpha f_\alpha * E\|_{L^p} \\ &\le \sum_{|\alpha|\le m} \|f_\alpha * \partial^\alpha E\|_{L^p} \\ &\le \sum_{|\alpha|\le m} \|f_\alpha\|_{L^p} \|\partial^\alpha E\|_{L^1(\mathbb{R}^n_x)}. \end{aligned} \tag{2.5}$$

By virtue of (2.2) we can obtain

$$\begin{aligned} \|\partial^\alpha E(x,t)\|_{L^1(\mathbb{R}^n_x)} &\le C^{|\alpha|} t^{-(|\alpha|+n)/2} \alpha!^{\frac{1}{2}} \left|\left| \exp\left(-\frac{|x|^2}{8t}\right) \right|\right|_{L^1(\mathbb{R}^n_x)} \\ &\le C_1 t^{-N} \end{aligned}$$

for some $C_1 > 0$ and $N > 0$. Combining this and (2.5) we have

$$\|U(x,t)\| \leq Ct^{-N} \quad \text{in} \quad \mathbb{R}^{n+1}_+$$

for another constant $C > 0$.

Now we prove the converse. To do this we construct continuous functions $v(t)$ and $w(t)$ on the real line such that

$$\left(\frac{d}{dt}\right)^m v(t) = \delta(t) + w(t), \qquad (2.6)$$
$$\operatorname{supp} v \subset [0,2], \quad \operatorname{supp} w \subset [1,2].$$

where m is an integer and $\delta(t)$ is the Dirac measure.

For example, consider $f(t)$

$$f(t) = \begin{cases} t^m & t \geq 0 \\ 0 & t < 0. \end{cases}$$

By multiplying a suitable cut off function we can obtain $v(t)$ and $w(t)$.

Define

$$G(x,t) = \int_0^\infty U(x,t+s)v(s)ds$$

and

$$H(x,t) = -\int_0^\infty U(x,t+s)w(s)ds$$

if we take $m \geq N+1$ in (2.6) it follows that $G(x,t)$ and $H(x,t)$ are continuous and bounded functions on $\mathbb{R}^n \times \{t \geq 0\}$. Moreover, in $\mathbb{R}^{n+1}_+$,

$$(\partial_t - \Delta)G(x,t) = 0,$$
$$(\partial_t - \Delta)H(x,t) = 0,$$

and

$$\left(\frac{d}{dt}\right)^m G(x,t) = \Delta^m G(x,t) \qquad (2.7)$$
$$= U(x,t) - H(x,t).$$

Applying the Hölder inequality we obtain for $\phi \in L^q$

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^M} |U(x,t+s)\phi(x)v(s)|dxds\\ &\le \int_0^2 \|U(x,t+s)\|_{L^p}\|\phi\|_{L^q}|v(s)|ds\\ &\le C\|\phi\|_{L^q}\end{aligned}$$

for some $C > 0$. Then it follows from Fubini theorem that $G(x,t)$ is a bounded linear functional on L^q, which means that $G(x,t) \in L^p(\mathbb{R}^n_x)$. Similarly, we can easily show that $H(x,t) \in L^p(\mathbb{R}^n_x)$. Then by the well known theorem (see [W, p.153]) there exist a couple of function $g(x), h(x) \in L^p$ such that

$$G(x,t) = g(x) * E(x,t), \quad H(x,t) = h(x) * E(x,t).$$

In fact, since $G(x,0) = g(x)$ and $H(x,0) = h(x)$ it follows that $g(x)$ and $h(x)$ are also continuous bounded function on $\mathbb{R}^n$.

Now define

$$u(x) = \Delta^m g(x) + h(x). \tag{2.8}$$

Then u obviously belongs to $\mathcal{D}'_{L^p}$ and we have

$$\begin{aligned}u * E &= (\Delta^m g + h) * E\\ &= \Delta^m(g * E) + h * E\\ &= \Delta^m G(x,t) + H(x,t)\\ &= U(x,t)\end{aligned}$$

and $U(x,t) \to u$ in $\mathcal{D}'_{L^p}$ as $t \to 0+$, which completes the proof.

In the proof of the above theorem, in fact, we can obtain the following structure theorem by (2.8):

THEOREM 2.4. *Let $u \in \mathcal{D}'_{L^p}, 1 < p < +\infty$. Then there exist a constant m and a couple of bounded continuous functions $g(x)$ and $h(x)$ in $L^p(\mathbb{R}^n)$ such that*

$$u(x) = \Delta^m g(x) + h(x).$$

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