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SYMBOLIC DYNAMICS AND UNIFORM DISTRIBUTION MODULO 2

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1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a measure space with the σ-algebra $\mathcal{B}$ and the probability measure μ. Throughout this article set equalities and inclusions are understood as being so modulo measure zero sets. A transformation T defined on a probability space X is said to be *measure preserving* if $\mu(T^{-1}E) = \mu(E)$ for $E \in \mathcal{B}$. It is said to be *ergodic* if $\mu(E) = 0$ or 1 whenever $T^{-1}E = E$ for $E \in \mathcal{B}$. Consider the sequence $\{x, Tx, T^2x, \dots\}$ for $x \in X$. One may ask the following question: What is the relative frequency of the points $T^n x$ which visit the set E? Birkhoff Ergodic Theorem states that for an ergodic transformation T the time average $\lim_{n\to\infty}(1/N)\sum_{n=0}^{N-1} f(T^n x)$ equals for almost every x the space average $(1/\mu(X))\int_X f(x)d\mu(x)$. In the special case when f is the characteristic function χ_E of a set E and T is ergodic we have the following formula for the frequency of visits of T-iterates to E:

$$\lim_{N\to\infty} \frac{|\{n : T^n x \in E, 0 \le n < N\}|}{N} = \mu(E)$$

for almost all $x \in X$ where $|\cdot|$ denotes cardinality of a set. For the details, see [8], [10].

Veech [9] first studied the following problem: Let

$$y_n(x) \equiv \sum_{k=0}^{n-1} \chi_E(T^k x) \pmod 2,\ y_n = 0, 1.$$

The problem is whether $\frac{1}{N}\sum_{n=1}^{N} y_n(x)$ converges in a suitable sense and, if so, what is the value. One might expect that its limit exists and equals

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$\frac{1}{2}$. Contrary to our intuition, it might not be equal to $\frac{1}{2}$ if $\exp(\pi i \chi_E)$ is a *coboundary*, that is, $\exp(\pi i \chi_E(x)) = \overline{q(x)} q(Tx)$ for some $q(x)$ of modulus 1. He showed that for any irrational rotation on $[0, 1)$ there exists an interval E such that the limit does not exist. Thus in this paper we also restrict our interest to the case when the measurable set E is an interval. There are some examples of coboundary. For the rotation by irrational θ, if E is any interval of length 2θ, then $\exp(\pi i \chi_E(x))$ is a coboundary. And if T is the homomorphism on $[0, 1)$ such that $x \to 2x \pmod 1$ and E is either $[1/6, 5/6]$ or $[1/4, 3/4]$, then $\exp(\pi i \chi_E(x))$ is a coboundary.

The problem can be changed into the following. Define $U : L^2(\mu) \to L^2(\mu)$ by $(Uf)(x) = \exp(\pi i \chi_E(x)) f(Tx)$. Then U is a linear isometry and

$$(U^n f)(x) = \exp(\pi i \sum_{k=0}^{n-1} \chi_E(T^k x)) f(T^n x).$$

It follows that

$$(U^n 1)(x) = \exp(\pi i \sum_{k=0}^{n-1} \chi_E(T^k x)) = \exp(\pi i y_n(x)).$$

Now our problem is expressed as the formula

$$\frac{1}{N} \sum_{n=1}^{N} (U^n 1)(x).$$

Here if $y_n(x) = 1$ then $U^n 1(x) = -1$ and if $y_n(x) = 0$ then $U^n 1(x) = 1$. Thus we ask whether the limit of the above formula is 0.

The Mean Ergodic Theorem states the following: For a bounded linear operator V on a Hilbert space $\mathcal{H}$ such that $\|V\| \leq 1$, let $\mathcal{H}_1 = \{f \in \mathcal{H} : Vf = f\}$, and let P be the orthogonal projection onto $\mathcal{H}_1$. Then $\frac{1}{N} \sum_{n=0}^{N-1} V^n f \to Pf$ in $\mathcal{H}$. For the proof, see [8].

PROPOSITION 1. *Let T be an ergodic transformation on X and let $(Uf)(x) = A(x) f(Tx)$ where $A(x)$ is real-valued and $|A(x)| = 1$. Put $\mathcal{M} = \{h \in L^2(X) : Uh = h\}$. Then*

(i) $\dim \mathcal{M} = 0$ *or* 1.

(ii) *If* $\dim \mathcal{M} = 0$ *then* $\frac{1}{N}\sum_{n=1}^{N} U^n 1 \to 0$ *in* $L^2(X)$.
(iii) *If* $\dim \mathcal{M} = 1$ *then*
  (a) $A(x)$ *is a coboundary.*
  (b) *there exists* $p(x) \in L^2(X)$ *such that* $p(x)$ *is real-valued,* $|p(x)| = 1$ *a.e. such that* $A(x) = \overline{p(x)}p(Tx)$.
  (c) *for the real valued* $p(x)$ *of (b), at a.e.* x *the sequence* $\frac{1}{N}\sum_{n=1}^{N} U^n 1$ *converges pointwise to* $p(x)\int_X p(t)\,dm(t)$.

Proof. Part (i) follows from the ergodicity of T and Part (ii) from the Mean Ergodic Theorem. For (iii),(a) let q be such that $Uq = q$. By definition, $A(x)q(Tx) = q(x)$, hence $A(x) = \overline{q(x)}q(Tx)$. Since $\overline{q^2(x)}q^2(Tx) = 1$, $q^2(Tx) = q^2(x)$, q^2 is constant, which we may choose to be equal to 1. Thus (b) is proved. In (c) of (iii) the Birkhoff Ergodic Theorem implies that at a. e. point $x \in X$

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}(U^n 1)(x) &= \frac{1}{N}\sum_{n=1}^{N} p(x)p(T^n x) \\ &= p(x)\frac{1}{N}\sum_{n=1}^{N} p(T^n x) \to p(x)\int_X p(y)\,d\mu(y).\end{aligned}$$

In [6] it was shown that the convergence theorem is in L^2. The pointwise convergence was not indicated there. Note that if $p = \pm 1$, then there exists a measurable subset F such that $p(x) = \exp(\pi i \chi_F(x))$, hence $\exp(\pi i \chi_E)$ is a coboundary if and only if $E = F \triangle T^{-1}F$ modulo measure zero sets where $\triangle$ is the symmetric difference. Hence for subsets E of the form $E = F\triangle T^{-1}F$ we do not have uniform distribution modulo 2 unless the measure of F equals 1/2. For coboundaries arising from irrational rotations, see [5].

2. The interval maps and symbolic dynamics

In this article we consider the behavior of the iterates of a map $f(x)$ of the unit interval to itself. We also assume that $f(x)$ is noninvertible and piecewise continuous. The central topic for this category of maps is to prove the existence and ergodicity of an f-invariant measure μ equivalent to the Lebesgue measure m while the equivalence meaning that μ and

m have the same sets of measure zero. There are several examples: (a) $f(x) = \{2x\}$ where $\{\cdot\}$ is the fractional part; (b) $f(x) = \{\beta x\}$, $\beta = (1+\sqrt{5})/2$; (c) the gauss transform $f(x) = \{1/x\}$. For these transformations explicit formulas are known for invariant measures: (a) the Lebesgue measure dx; (b) $d\mu = \beta dx$, $0 \le x < \beta^{-1}$, and $d\mu = dx$, $\beta^{-1} < x \le 1$; (c) the Gauss measure $d\mu = (1/\log 2)dx/(1+x)$.

The Gauss transformation is also called the continued fraction map. Define

$$[n_1, \ldots, n_k] = \cfrac{1}{n_1 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}, \quad [n_1, n_2, \ldots] = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}}$$

where n_i's are positive integers. Since $[n_1, \ldots, n_{k-1}, 1] = [n_1, \ldots, n_{k-1} + 1]$, we shall use an expression $[n_1, \ldots, n_{k-1}+1]$ instead of $[n_1, \ldots, n_{k-1}, 1]$. The continued fraction can be expressed using the Gauss transformation: Let $a(x)$ denote an integral part of $1/x$ where $x \in (0,1)$. If $x = [n_1, n_2, n_3, \ldots]$ then $a(f^k x) = n_{k+1}$. So f is nothing but a shift on infinitely many symbols. For continued fractions theory see [7].

FACT. *Let $f : [0,1] \to [0,1]$ and let $\{I_i\}$ be a finite partition of $[0,1]$. Assume that f satisfies*

(1) *$f|I_i$, the restriction of f to I_i, has a C^2-extension to the closure $\overline{I}_i$ of I_i.*
(2) *$f|I_i$ is strictly monotone.*
(3) *$f(I_i) =$ union of some $\overline{I}_j$'s.*
(4) *There exists an integer p such that $f^p(\overline{I}_i) = [0,1]$ for all i.*

If f is eventually expansive, that is, for some iterate f^n, $|df^n/dx| \ge \theta > 1$ for all x, then f has a finite Lebesgue equivalent measure μ and is ergodic. Furthermore $d\mu = \rho(x)dx$ where $\rho(x)$ is piecewise continuous and $1/D < \rho < D$ for some $D > 0$.

To apply the result to Gauss transformation we need to modify the conditions. Since we are interested in symbolic dynamics arising from a finite alphabet, it will not be studied here. For the details see [1], [2].

Let $T : C \to C$ be an invertible transformation. By partitioning C into $C = \bigcup_{i=1}^{N} C_i$ we can associate to every point $p \in C$ a two-sided infinite symbolic sequence of labels $[\ldots, x_{n-1}, x_n, x_{n+1}, \ldots]$ where the

x_n assume the labels of C_i according as $T^n p \in C_{x_n}$, $n \in \mathbb{Z}$. We let Σ be the space of these symbolic sequences. Acting on the group of iterates of the shift transformation σ defined by $\sigma[\ldots, x_{n-1}, x_n, x_{n+1}, \ldots] = [\ldots, y_{n-1}, y_n, y_{n+1}, \ldots]$ where $y_n = x_{n+1}$.

The baker's transformation T is defined on the unit square by $T(x, y) = (2x, \frac{1}{2}y)$ for $0 \le x < \frac{1}{2}$ and $T(x, y) = (2x - 1, \frac{1}{2}y + \frac{1}{2})$ for $\frac{1}{2} \le x \le 1$. It is nothing but the shift transformation on the full shift space of all the two-sided infinite sequences consisting of 0's and 1's.

In this article we study non-invertible map. So the corresponding symbolic systems are one-sided. Let $f : X \to X$ be a non-invertible map. By partitioning X into $X = \bigcup_{i=1}^{N} I_i$ we associate to $p \in X$ an f-expansion, i.e., a one-sided infinite sequence of symbols $[x_0, x_1, \ldots, x_n, \ldots]$ where x_n are determined by $f^n p \in I_{x_n}$, $n = 0, 1, 2, \ldots$. On the space Σ^+ of f-expansions we have the one-sided shift transformation, again denoted by σ, defined by $\sigma[x_0, x_1, \ldots] = [x_1, x_2, \ldots]$. The transformation $x \mapsto \{2x\}$ is identical with the one-sided shift on Σ^+ consisting of one-sided infinite sequences of symbols $0, 1$: To a point $p = \sum_n x_n 2^{-n}$, $x_n = 0, 1$, there corresponds a sequence $[x_1, x_2, \ldots]$. Here we exclude a subset of measure zero to identify two systems.

Let T be the map $x \mapsto \{2x\}$ that is ergodic with respect to the Lebesgue measure. Let S be the shift on $\prod_{n=1}^{\infty}\{0, 1\}$ that is ergodic with respect to the Bernouilli measure. From the measure theoretic point of view two transformations T and S are identical. Two transformations T_1 and T_2 on probability spaces X_1 and X_2 are said to be *isomorphic* if there exists a measure preserving transformation $\phi : X_1 \to X_2$ which is one-to-one almost everywhere satisfying $\phi \circ T_1 = T_2 \circ \phi$ a.e. on X_1. For example, T and S are isomorphic.

A rational number x of the form $x = \sum_{k=1}^{n} a_k 2^{-k}$, $a_k = 0, 1$, is called a *binary fractional number.* In [6] it was proved that if I is a subinterval in $[1/2, 1)$ with binary fractional endpoints then $\exp(\pi i \chi_I)$ is not a coboundary with respect to the transformation $T : x \mapsto 2x$. For the isomorphic transformation S, we obtain the following corresponding result: If a set I is a cylinder set in $\prod_{n=1}^{\infty}\{0, 1\}$ of the form $I = \{1\} \times C_2 \times \cdots \times C_k \times \prod_{n=k+1}^{\infty}\{0, 1\}$, where C_n's are subsets of $\{0, 1\}$, then there is no measurable set E such that $I = E \triangle T^{-1} E$. We will show that for any suitable transformation on the unit interval that gives rise to a shift space on a finite alphabet the same conclusion holds true. In the following construction we consider perturbations of the map $x \mapsto \{2x\}$.

DEFINITION 1. Let $f : [0,1] \to [0,1]$ and let $\{I_0, I_1\}$ be a partition of $[0,1]$ where I_1 and I_2 are intervals of positive length. Assume that f satisfies

(1) $f|I_i$, the restriction of f to I_i, has a C^2-extension to the closure $\overline{I_i}$ of I_i.
(2) $f|I_i$ is strictly increasing.
(3) $f(\overline{I_i}) = [0,1]$.

And suppose that for some n, $|df^n/dt| \geq \theta > 1$ for all t. If we regard the above map f as being defined on the unit circle since $f(1) = 1$, its winding number equals 2. We call it a *generalized angle doubling map*.

DEFINITION 2. Given a generalized angle doubling map f, construct a shift space on two symbols $0, 1$ as follows: To each $t \in [0,1]$ there corresponds a one-sided infinite sequence $[x_0, x_1, \ldots, x_n, \ldots]$ such that $f^n(t) \in I_{x_n}$. There is a possibility that for a point there might exist more than one representations as a sequence but these points form a countable set that has measure zero for any continuous measure. So we ignore them. Let $X = \prod_{n=1}^{\infty}\{0,1\}$ be the set of all such sequences. Since f has a finite Lebesgue equivalent ergodic measure $\rho(t)\,dt$, we define a shift invariant measure ν on a cylinder set $C = C_1 \times \cdots \times C_N \times \prod_{n=N+1}^{\infty}\{0,1\} \subset X$ by $\mu(C) = \int_E \rho(t)dt$ where $E = \{t \in [0,1] : t = \sum_{n=1}^{\infty} a_n 2^{-n}, a_n \in C_n, 1 \leq n \leq N\}$. Note that E is a union of intervals with binary fractional endpoints. Kolmogorov Extension Theorem guarantees the existence and uniqueness of such a measure μ. We call the shift space X the *binary symbolic system obtained from* f and use the notation X_f if necessary. By construction f and σ are isomorphic.

REMARK. Our construction does not apply if the condition (3) does not hold. For example, the interval map $x \mapsto \beta x$, $\beta = (1+\sqrt{5})/2$ has the following special property: Put $I_0 = [0, \beta^{-1}]$, $I_1 = [\beta^{-1}, 1]$. If $x \in I_1$, then $Tx \in I_0$. In other words, in any sequence $[x_0, x_1, \ldots, x_n, \ldots]$ the symbol 1 does not occur consecutively. Hence in this case X_f would not be the full shift $\prod_1^{\infty}\{0,1\}$. In fact, it a shift of finite type with a forbidden block $\{11\}$. For shifts of finite type, see [1].

3. Uniform distribution modulo 2

DEFINITION 3. A *binary multi-index* $\vec{n}$ is a finite sequence of 0,1 and denoted as $\vec{n} = (n_1, \ldots, n_k)$ and k is called the *length* of $\vec{n}$ and denoted by $|\vec{n}|$. If there is no danger of ambiguity we call it a *multi-index*. If $|\vec{n}| = 1$, i.e., $\vec{n} = (n_0)$ for some n_0, then we write $\vec{n} = n_0$.

Let f be a map as given in Definition 1. Define $g_i : [0,1] \to [0,1]$ by letting $g_i(t)$ be the unique element in the set $f^{-1}(\{t\}) \cap I_i$, $i = 0, 1$. Note that $f^{-1}(\{t\}) = \{g_0(t), g_1(t)\}$. For a multi-index $\vec{n} = (n_1, \ldots, n_k)$, define $g_{\vec{n}} = g_{n_1} \circ \cdots \circ g_{n_k}$.

The following propositions are obtained easily from the definitions. Their proofs are skipped.

PROPOSITION 2. *Put $\vec{n} = (n_1, \ldots, n_k)$. Let f and g_i be as given in Definition 1. Define $[n_1, \ldots, n_k]$ to be the point $g_{\vec{n}}(0)$. In the case $f(x) = \{2x\}$ this yields a rational number of the form $p/2^k$, $0 \le p < 2^k$. Then we have the following*

(1) $f^{-k}(\{t\}) = \{g_{\vec{n}}(t) : |\vec{n}| = k\}$.
(2) $f^k(g_{\vec{n}}(x)) = x$ *where* $k = |\vec{n}|$.
(3) $g_{\vec{n}}$*'s are strictly increasing.*
(4) $g_{\vec{n}}([a_1, a_2, \ldots]) = [n_1, \ldots, n_k, a_1, a_2, \ldots]$.
(5) $f^{-k}(E) = \bigcup_{|\vec{n}|=k} g_{\vec{n}}(E)$ *for any subset* E.

PROPOSITION 3. *For any fixed integer $k > 0$ let $\mathcal{P}_k$ be the set of numbers of the form $[a_1, \ldots, a_k]$ where $a_i = 0, 1$. Then the points in $\mathcal{P}_k$ partition the whole interval $[0,1]$ into 2^k segments. And we have*

(1) $g_{\vec{n}}([0,1])$ *is an interval with its endpoints from the set* $\mathcal{P}_k$, $k = |\vec{n}|$.
(2) *If* $x \in g_{\vec{n}}([0,1))$, *then* $x = [n_1, \ldots, n_k, \ldots]$ *for* $\vec{n} = (n_1, \ldots, n_k)$.
(3) *We give an ordering on $\bigcup_{k=1}^{\infty} \mathcal{P}_k \subset [0,1]$ by using the usual ordering on the real line. Since $f|_{I_i}$ is strictly increasing, the map $\phi_f : \bigcup_{k=1}^{\infty} \mathcal{P}_k \to X_f$ defined by $\phi_f([a_1, \ldots, a_k]) = g_{[a_1, \ldots, a_k]}(0)$ is order-preserving where X_f has the usual dictionary ordering.*

Let $\sigma : X \to X$ be the usual Bernouilli shift on two symbols 0,1 with probabilities $1/2$ on each and let τ be the angle doubling map $x \mapsto \{2x\}$.

The following diagram is commutative:

$$
\begin{array}{ccc}
([0,1], \rho\, dt) & \xrightarrow{\ f\ } & [0,1] \\
{\scriptstyle \phi_f}\big\downarrow & & {\scriptstyle \phi_f}\big\downarrow \\
(X_f, d\nu_f) & \xrightarrow{\ \sigma_f\ } & X_f \\
{\scriptstyle \psi}\big\downarrow & & {\scriptstyle \psi}\big\downarrow \\
(X, d\nu) & \xrightarrow{\ \sigma\ } & X \\
{\scriptstyle \phi}\big\uparrow & & {\scriptstyle \phi}\big\uparrow \\
([0,1], dt) & \xrightarrow{\ \tau\ } & [0,1]
\end{array}
$$

The following result shows that uniform distribution mod 2 is stable under small perturbations.

THEOREM. *Let f be a generalized angle doubling map where the partition of $[0,1]$ is given by $\{I_0, I_1\}$ as in Definition 1. Then it has the same uniform distribution mod 2 property as the angle doubling map $\tau : x \mapsto \{2x\}$. That is, an interval I with endpoints from $\bigcup_{k=1}^{\infty} \mathcal{P}_k$ satisfies $I = E \triangle f^{-1} E$ for some measurable set E if and only if $(\phi^{-1} \circ \psi \circ \phi_f)(I) = F \triangle \tau^{-1} F$ for some measurable set F.*

REMARK. The idea is that the set $(\phi^{-1} \circ \psi \circ \phi_f)(I)$ is an interval with binary fractional endpoints since f is piecewise strictly increasing. Hence for such an interval we have uniform distribution modulo 2.

Proof. Suppose that there exists an interval I satisfying the conditions in the theorem. Let E be the set such that $\int_{I \triangle (E \triangle f^{-1} E)} \rho(t) dt = 0$. From Propositions 2 and 3 it is easy to see that (i) $\phi^{-1} \circ \psi \circ \phi_f(I)$ is an interval with binary fractional endpoints, and (ii) a set A has measure zero with respect to $\rho\, dt$ if and only if $(\phi^{-1} \circ \psi \circ \phi_f)(A)$ has measure zero with respect to the Lebesgue measure.

COROLLARY. *Let f be a generalized angle doubling map where the partition of $[0,1]$ is given by $\{I_0, I_1\}$. If an interval $[a,b]$ satisfies (i) $a < b$, (ii) $a, b \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$, and (iii) $[a,b] \subset I_1$, then there is no measurable* *subset E such that $[a,b] = E\triangle f^{-1}E$. Hence for such an interval we have uniform distribution modulo 2.*

Proof. Suppose that there exists an interval $[a,b]$ satisfying the conditions. Let E be the set such that $\int_{[a,b]\triangle(E\triangle f^{-1}E)} \rho(t)dt = 0$. From Propositions 2 and 3 it is easy to see that (i) $\phi^{-1} \circ \psi \circ \phi_f([a,b])$ is an interval with binary fractional endpoints, and (ii) a set A has measure zero with respect to $\rho\, dt$ if and only if $(\phi^{-1} \circ \psi \circ \phi_f)(A)$ has measure zero with respect to the Lebesgue measure. Now let $A = [a,b]\triangle(E\triangle f^{-1}E)$ and apply the result in [6] which states that the theorem holds true for the map $x \mapsto \{2x\}$.

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