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ON QUASIAFFINE TRANSFORMS OF QUASISUBSCALAR OPERATORS

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1. Introduction

In this paper we characterize the quasiaffine transforms of quasisubscalar operators.

Let $\mathbf{H}$ and $\mathbf{K}$ be separable, complex Hilbert spaces and $\mathcal{L}(\mathbf{H}, \mathbf{K})$ denote the space of all linear, bounded operators from $\mathbf{H}$ to $\mathbf{K}$. If $\mathbf{H} = \mathbf{K}$, we write $\mathcal{L}(\mathbf{H})$ in place of $\mathcal{L}(\mathbf{H}, \mathbf{K})$. A linear bounded operator S on $\mathbf{H}$ is called scalar of order m if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(\mathbf{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbf{C}$, and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. We now define the weaker form of a subscalar operator. An operator T in $\mathcal{L}(\mathbf{K})$ is quasisubscalar if there exists a one-to-one V in $\mathcal{L}(\mathbf{K}, \mathbf{H})$ such that $VT = SV$ where S is a scalar operator.

Let us define now a special Sobolev type space. Let U be a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The vector valued Sobolev space $W^m(U, \mathbf{H})$ with respect to $\bar{\partial}$ and order m will be the space of those functions f in $L^2(U, \mathbf{H})$ whose derivatives $\bar{\partial} f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathbf{H})$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \|\bar{\partial}^i f\|_{2,U}^2$$

$W^m(U, \mathbf{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathbf{H})$.

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An X in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator S in $\mathcal{L}(\mathbf{H})$ is said to be a quasiaffine transform of an operator T in $\mathcal{L}(\mathbf{K})$ if there is a quasiaffinity X in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ such that $XS = TX$ (notation; $S \prec T$). The paper has been divided into three sections. Section two deals with quasiaffine transforms of hyponormal operators. In section three, we study quasiaffine transforms of quasisubscalar operators.

2. Quasiaffine transforms of hyponormal operators

An operator W is called a unilateral weighted shift if there is an orthonormal basis $\{e_n : n \geq 0\}$ and a sequence of scalars $\{\alpha_n\}$ such that $We_n = \alpha_n e_{n+1}$ for all $n \geq 0$. It is not hard to note that W is bounded if and only if $\{\alpha_n\}$ is bounded. Bilateral weighted shifts are defined analogously.

PROPOSITION 2.1([Co 2, PROPOSITION 8.6]). *A weighted shift is hyponormal if and only if its weight sequence is increasing.*

The following example shows that quasiaffinity does not preserve hyponormality.

EXAMPLE 2.2. Let $\mathbf{H}$ be a separable Hilbert space and let $\{e_n\}$ be an orthonormal basis of $\mathbf{H}$. Define a weighted shift W on $\mathbf{H}$ as follows;

$$We_0 = e_1,\ We_1 = \sqrt{2}e_2,\ We_n = e_{n+1} \text{ for all } n \geq 2.$$

Then there exists an operator X in $\mathcal{L}(\mathbf{H})$ such that $Xe_0 = e_0$, $Xe_1 = e_1$, and $Xe_n = \frac{1}{\sqrt{2}}e_n$ for all $n \geq 2$. It is clear that X is a quasiaffinity, and that $XW = SX$ where S is the unilateral shift. But from Proposition 2.1, W is not hyponormal.

PROPOSITION 2.3. *Let T in $\mathcal{L}(\mathbf{H})$ be hyponormal, and let A be any operator in $\mathcal{L}(\mathbf{H})$ such that $A \prec T$. Then $\ker(A - z) = \ker(A - z)^2$ for every $z \in \mathbf{C}$.*

Proof. Let $z \in \mathbf{C}$. Since $(T - z)(T - z)^* \leq (T - z)^*(T - z)$, there is an operator $K_z \in \mathcal{L}(\mathbf{H})$ (see [Do]) such that

$$T - z = (T - z)^* K_z.$$

Since $(T-z)^* = K_z^*(T-z)$,

$$\ker(T-z) \subseteq \ker(T-z)^*.$$

If $x \in \ker(T-z)^2$, then $(T-z)x \in \ker(T-z)$. Therefore, $(T-z)x \in \ker(T-z)^*$.
That implies

$$\begin{aligned}\|(T-z)x\|^2 &= \langle (T-z)^*(T-z)x, x\rangle \\ &\leq \|(T-z)^*(T-z)x\|\|x\| = 0.\end{aligned}$$

Thus $x \in \ker(T-z)$. That is, $\ker(T-z)^2 \subseteq \ker(T-z)$. Trivially $\ker(T-z) \subseteq \ker(T-z)^2$, so $\ker(T-z) = \ker(T-z)^2$. Now we want to show that $\ker(A-z) = \ker(A-z)^2$. Clearly, $\ker(A-z) \subseteq \ker(A-z)^2$. If $x \in \ker(A-z)^2$, then $(A-z)^2x = 0$. Let X be a quasiaffinity such that $XA = TX$. Then $X(A-z)^2x = 0$. Since $XA = TX$, $(T-z)^2Xx = 0$. Therefore, $Xx \in \ker(T-z)^2 = \ker(T-z)$. Since $(T-z)Xx = 0$, $X(A-z)x = 0$. Since X is one-to-one, $(A-z)x = 0$. Therefore, $x \in \ker(A-z)$.

DEFINITION. An operator T in $\mathcal{L}(\mathbf{H})$ is said to satisfy the single valued extension property if for any open subset U in $\mathbf{C}$, the function

$$z - T : O(U, \mathbf{H}) \longrightarrow O(U, \mathbf{H})$$

defined by the obvious point wise multiplication is one-to-one where $O(U, \mathbf{H})$ denotes the Fréchet space of $\mathbf{H}$-valued analytic functions on U with respect to norm topology. If, in addition, the above function $z - T$ has closed range on $O(U, \mathbf{H})$, then T satisfies the Bishop's condition (β).

In other terms, condition (β) means that, for any open set U, and any sequence of analytic functions $f_n \in O(U, \mathbf{H})$, $\lim_{n\to\infty} f_n = 0$ in $O(U, \mathbf{H})$ whenever $\lim_{n\to\infty}(z-T)f_n = 0$. In particular, $(z-T)g = 0$ if and only if $g = 0$, where $g \in O(U, \mathbf{H})$.

LEMMA 2.4([MP, THEOREM 5.5]). *Every hyponormal operator has property (β).*

PROPOSITION 2.5. *Let $T \in \mathcal{L}(\mathbf{H})$ be hyponormal, and let A be any operator in $\mathcal{L}(\mathbf{H})$ such that $A \prec T$. Then A has the single valued extension property.*

Proof. For any open subset U of $\mathbf{C}$, let $f \in O(U, \mathbf{H})$ such that $(z - A)f = 0$. Let X be a quasiaffinity such that $XA = TX$. Since X is one-to-one, $X(z - A)f = 0$. Therefore, $(z - T)Xf = 0$. Since T has the single valued extension property by Lemma 2.4 and X is one-to-one, $f = 0$.

3. Quasiaffine transforms of quasisubscalar operators

THEOREM 3.1. *Let $T \in \mathcal{L}(\mathbf{H})$ be quasisubscalar, i.e., There exists an one-to-one V of $\mathbf{H}$ into $H(D)$ defined by $1 \otimes h + \overline{(z - T)W^m(D, \mathbf{H})}$ such that $VT = \tilde{M}_z V$ where $H(D) = \dfrac{W^m(D, \mathbf{H})}{cl(z - T)W^m(D, \mathbf{H})}$, M_z is the multiplication operator on $W^m(D, \mathbf{H})$, and $\tilde{M}_z$ is the class of M_z on $H(D)$. Let $A \in \mathcal{L}(\mathbf{H})$ be any operator such that $XA = TX$ where X is one-to-one. Then A is quasisubscalar.*

REMARK. Since V and X are one-to-one, VX is one-to-one. Therefore, VX implements the quasisubscalar properties. But we shall use Putinar's technique to prove Theorem 3.1.

LEMMA 3.2. $\overline{ran(z - A)} \neq W^m(D, \mathbf{H})$.

Proof. If not, $\overline{\text{ran}(z - A)} = W^m(D, \mathbf{H})$. Since $XA = TX$, $X(\text{ran}(z - A)) \subset \text{ran}(z - T)$. That implies $X(\overline{\text{ran}(z - A)}) \subset \overline{\text{ran}(z - T)}$. From the hypothesis, $XW^m(D, \mathbf{H}) \subset \overline{\text{ran}(z - T)}$. Therefore, $VXh = \tilde{0}$ for any nonzero h in $\mathbf{H}$. Since V and X are one-to-one, $h = 0$. So we have a contradiction.

Proof of Theorem 3.1. Since $\overline{\text{ran}(z - A)} \neq W^m(D, \mathbf{H})$ from Lemma 3.2, there is a non-zero

$$\Gamma : \mathbf{H} \longrightarrow \frac{W^m(D, \mathbf{H})}{cl(z - A)W^m(D, \mathbf{H})}$$

defined by $\Gamma h = 1 \otimes h + \overline{(z - A)W^m(D, \mathbf{H})}$. It is enough to show that Γ is one-to-one. If not, there exists a non-zero $h \in \mathbf{H}$ such that $\Gamma h = \tilde{0}$, *i.e.*, $h \in \overline{(z-A)W^m(D, \mathbf{H})}$. Therefore, there exists a sequence $\{g_n\} \in W^m(D, \mathbf{H})$ such that

$$\lim_{n\to\infty} \|(z-A)g_n - h\|_{W^m} = 0.$$

That implies

$$\lim_{n\to\infty} \|X(z-A)g_n - Xh\|_{W^m} = 0.$$

Since $XA = TX$,

$$\lim_{n\to\infty} \|(z-T)Xg_n - Xh\|_{W^m} = 0.$$

Since $\{Xg_n\} \in W^m(D, \mathbf{H})$, $Xh \in \overline{\operatorname{ran}(z-T)}$. Therefore, $VXh = \tilde{0}$. Since V and X are one-to-one, $h = 0$. So we have a contradiction. Thus Γ is one-to-one. Therefore, A is quasisubscalar.

COROLLARY 3.3. *Let $T \in \mathcal{L}(\mathbf{H})$ be hyponormal, and let S be any bounded linear operator such that $XS = TX$ where X is one-to-one. Then S is quasisubscalar.*

Proof. Since T is subscalar by [Pu], it follows from Theorem 3.1.

COROLLARY 3.4. *If X is bounded below in Corollary 3.3, then S is a non-hypo-*
normal subscalar operator of order 2.

REMARK. Corollary 3.3 implies that W in example 2.2 is non-hyponormal quasisubscalar.

The next result generalizes Lemma 2.4 in [Pu].

THEOREM 3.5. *Let $T \in \mathcal{L}(\mathbf{H})$ be hyponormal, and let S be any operator in $\mathcal{L}(\mathbf{H})$ such that $XS = TX$ where X is one-to-one. Then the operator*

$$z - S : W^2(D, \mathbf{H}) \longrightarrow W^2(D, \mathbf{H})$$

is one-to-one for an arbitrary bounded disk D in $\mathbf{C}$.

Proof. Let f be in $W^2(D, \mathbf{H})$ be such that $(z-S)f = 0$. Then $X(z-S)f = 0$. Since $XS = TX$, $(z-T)Xf = 0$. Since $z - T$ is one-to-one by Lemma 2.4, $Xf = 0$. Since X is one-to-one, $f = 0$. Thus $z - S$ is one-to-one.

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