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MANN-ITERATION PROCESS TO THE SOLUTION OF $y = x + Tx$ FOR AN ACCRETIVE OPERATOR T IN SOME BANACH SPACES

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1. Introduction

If H is a Hilbert space, then an operator $T : D(T) \subset H \to H$ is said to be monotone if

$$(x - y, Tx - Ty) \geq 0$$

for any x, y in $D(T)$.

Many authors [1], [4] obtained the existence theorems for the equation $y = x + Tx$ for x, given an element y in H and a monotone operator T. On the other hand some iterative methods were applied to the approximations for the solution of the above equation [6], [8]. For example Bruck [2] obtained the iterative solution of the above equation with an explicit error estimate as follows.

THEOREM. *Suppose T is a continuous single-valued monotone operator with open domain $D(T)$ in a Hilbert space H and $y \in R(I + T)$. Then there exists a neighborhood $N \subset D(T)$ of $\bar{x} = (I + T)^{-1}y$ such that for any initial guess $x_1 \in N$ the sequence generated from x_1 by*

$$x_{n+1} = \frac{n}{n+1}x_n - \frac{1}{n+1}(Tx_n - y)$$

remains in $D(T)$ and converges to $\bar{x}$ with estimate $\|x_n - \bar{x}\| = O(n^{-1/2})$.

Chidume [3] extended the above result of Bruck in L^p spaces for $p \geq 2$.

Let X be a Banach space. An operator $T : D(T) \subset X \to X$ is said to be accretive if

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\|$$

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for all x, y in $D(T)$ and $r > 0$. It is well-known that $T : D(T) \subset X \to X$ is accretive if

$$(Tx - Ty, J(x - y)) \geq 0$$

for all x, y in $D(T)$, where J is the normalized duality map of X. From this fact we know that an accretive operator in a Hilbert space is monotone.

X.Weng [8] generalized the result of Chidume for an accretive operator in uniformly smooth Banach space. In [8] by using the inequality of Reich [7] the author applied the Mann iteration method to approximate the solution of $y \in x + Tx$ for x, given an element y in uniformly smooth Banach space and an accretive operator T. In this note we show that the Mann iteration method can also be applied to the solution of the above equation in some Banach spaces. Our method of proof is based on the inequality [5] which holds in some Banach spaces. The inequality which is different from the Reich's also holds in uniformly smooth Banach space.

2. Main result

Let X be a Banach space. A Banach space X is called smooth if the norm of X is Gâteaux differentiable on $X - \{0\}$. The duality map J is defined by

$$J(x) = \{\, x^* \in X^* \mid (J(x), x) = \|x\|^2, \|J(x)\| = \|x\| \,\},$$

where X^* is the dual of X and $(\ ,\ )$ is the dual pairing. In a smooth Banach space J is single-valued. A Banach space $(X, \|\cdot\|)$ is called uniformly smooth if X^* is uniformly convex. In a uniformly smooth Banach space J is uniformly continuous on bounded subsets of X.

We need the following lemma which is well-known.

LEMMA 1[5]. *Let $\{\beta_n\}$ be a nonnegative real sequence and suppose $\{\beta_n\}$ satisfies the following inequality*

$$\beta_{n+1} \leq (1 - \alpha_n)\beta_n + \varepsilon\alpha_n,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \alpha_n \in (0, 1),\ \varepsilon > 0$. Then

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \varepsilon.$$

On the other hand, we also need the following lemma [5].

LEMMA 2[5]. *Let $(X, \|\cdot\|)$ be a smooth Banach space of X. Suppose one of the followings holds.*

(1) *J is uniformly continuous on any bounded subsets of X.*
(2) *$(J(x) - J(y), x - y) \leq \|x - y\|^2$, for all x, y in X.*
(3) *For any bounded subset D of X there is a nonnegative real-valued function c such that*

$$(J(x) - J(y), x - y) \leq c(\|x - y\|),$$

for all x, y in D where c satisfies $\lim_{t\to 0^+} c(t)/t = 0$.

Then for any $\varepsilon > 0$ and any bounded subset C there is $\delta > 0$ such that

$$\|tx + (1 - t)y\|^2 \leq 2(J(y), x)t + 2\varepsilon t + (1 - 2t)\|y\|^2,$$

for any $x, y \in C$ and $t \in [0, \delta)$.

REMARK. If X is uniformly smooth, then (1) in Lemma 2 holds. And if X is a Hilbert space, then (2) in Lemma 2 holds.

Now we state the main result.

THEOREM. *Let $(X, \|\cdot\|)$ be a smooth Banach space satisfying one of the assumptions in Lemma 2. Let $T : D(T) \subset X \to X$ be an accretive operator. Assume that $R(T)$ is bounded and $y \in R(I + T)$ and $\bar{x} = (I + T)^{-1}y$. If the following Mann iteration process*

(1) $x_1 \in D(T)$,
(2) $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(y - Tx_n)$,
(3) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \alpha_n \in [0, 1)$

could be defined, the sequence $\{x_n\}$ converges strongly to $\bar{x}$.

Proof. First we prove that $\{x_n\}$ is bounded. From

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq (1 - \alpha_n)\|x_n - \bar{x}\| + \alpha_n\|y - Tx_n - \bar{x}\| \\ &= (1 - \alpha_n)\|x_n - \bar{x}\| + \alpha_n\|T\bar{x} - Tx_n\|\end{aligned}$$

We have

$$\|x_{n+1} - \bar{x}\| \leq (1 - \alpha_n)\|x_n - \bar{x}\| + \alpha_n M,$$

where $M = \sup \{\|Tx - Ty\|; x, y \in D(T)\}$. From Lemma 1 $\{x_n\}$ is bounded. Let C be a bounded subset containing $R(T)$ and $\{x_n\} \cup \{\bar{x}\}$. For given $\varepsilon > 0$ and a bounded subset $C - C$, we have $\delta > 0$ in Lemma 2. Since $\lim_{n \to \infty} \alpha_n = 0$, there exists N such that for all $n \geq N, \alpha_n < \delta$. For such an n,

$$\begin{aligned} &\|x_{n+1} - \bar{x}\|^2 = \|\alpha_n(y - Tx_n - \bar{x}) + (1 - \alpha_n)(x_n - \bar{x})\|^2 \\ &\leq 2(J(x_n - \bar{x}), y - Tx_n - \bar{x})\alpha_n + 2\varepsilon\alpha_n + (1 - 2\alpha_n)\|x_n - \bar{x}\|^2 \end{aligned}$$

by Lemma 2.

Since T is accretive, the following holds.

$$\|x_{n+1} - \bar{x}\|^2 \leq 2\varepsilon\alpha_n + (1 - 2\alpha_n)\|x_n - \bar{x}\|^2.$$

In order to apply Lemma 1 we let $\beta_n = \|x_n - x\|^2$. Then

$$0 \leq \limsup_{n \to \infty} \beta_n \leq \varepsilon.$$

Since ε is arbitrary, $\limsup_{n \to \infty} \beta_n = 0$. So $\lim_{n \to \infty} \beta_n = 0$. The proof is complete.

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