Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 803–808

# SOME EQUIVALENTS OF LÖB'S THEOREM

S. M. KIM

Since Löb's announcement of his solution to Henkin's problem (Löb (1954, 1955)) there has been successful and fruitful research on provability logic tied up with modal logic. Specially, Löb's Theorem is of far-reaching significance in the following meta-mathematical and philosophical sense.

1. Löb's Theorem plus some additional properties of the modal box $\Box$ standing for provability axiomatize completely provability logic theory – cf. Solovay (1976,1985) for details.
2. Corresponding to this Theorem, Henkin discovered Löb's paradox underlying the sentence $\varphi$ with the biconditional

$$\varphi \leftrightarrow (Pr(\ulcorner \varphi \urcorner) \rightarrow \sigma).$$

3. Transcending Gödel's Second Incompleteness Theorem, Löb's Theorem as the solution to Henkin's problem and to the more general problem characterizes those instances of faithfulness provable in the theory as those trivially so provable – cf. Smorynski (1991).

As the proof of Gödel's Second Theorem is largely a formalization of the proof of Gödel's First Theorem, the Second Theorem might be said to have provided the first a cross check on proposed consistency proofs, despite Gödel's remark on the irrelevance of his Second Theorem to any sensible consistency problem. (As G.Kreisel in Kreisel (1993) cited: if Con(F) is in doubt, why should it be proved in F and not in an incomparable system?) The proofs of the provable equivalence between Gödel's Second Theorem and Löb's Theorem and the significant aspect

Received August 18, 1994. Revised September 29, 1994.

I would like to acknowledge DAAD grant. I also would like to thank the logicians at the university of Bonn as well as Otto Wilhelms Carls for discussions that inspired me to write this paper during the Summer 1993.

#1 above of Löb's Theorem show that Gödel's Second Theorem is of central importance for the topic of formal provability.

The present paper will be devoted to establishing the provable equivalence of Löb's Theorem and some other propositions involving Gödel's standard provability conditions with or without Faithfulness condition.

First however, we comment on notations. For the most part, our notation will be that commonly employed in mathematical logic. We write $F \vdash S$ to mean that "$S$ is a theorem of $F$". $\omega - con(\Sigma)$ stands for "$\Sigma$ is $\omega$- consistent."

We start with :

DEFINITION 1.

(1) $\omega = \{0, 1, 2, \dots\}$

(2) $\bot \overset{def}{=} (0 = f0)$ denoting logical falsehood, where $f$ is successor function.

(3) Following Gödel's procedure, a $\Sigma_1$ formula $Pr(x)$ (from "provable") can be constructed and be called a standard provability predicate iff it satisfies the following conditions : ( for all sentences $A$ and $B$ )
(Adequacy) $\vdash A \Rightarrow \vdash Pr(\ulcorner A \urcorner)$
(Conditionalization) $\vdash Pr(\ulcorner A \to B \urcorner) \to [Pr(\ulcorner A \urcorner) \to Pr(\ulcorner B \urcorner)]$
(Formal Adequacy) $\vdash Pr(\ulcorner A \urcorner) \to Pr(\ulcorner Pr(\ulcorner A \urcorner) \urcorner)$

(4) Condition : (Faithfulness) For any sentence $A$, $\vdash Pr(\ulcorner A \urcorner) \Rightarrow \vdash A$. Some authors call this condition reflection or soudness.

(5) Con $\overset{def}{=} \sim Pr(\ulcorner \bot \urcorner)$

Unless otherwise specified, let $Pr(x)$ be a standard provability predicate. The notation of the underlying axiomatic theory in question will be sometimes omitted when it is clear in the context.

FACT 2 (KREISEL / LÉVY (1968)).
*Löb's Theorem(LT) $\Rightarrow$ Gödel's Second Incompleteness Theorem (G2)*

*Proof.*
G2 is the special case of LT : $\vdash (Pr(\ulcorner A \urcorner) \to A) \Rightarrow \vdash A$, when $A = \bot$

FACT 3 (KRIPKE (1967) : SMORYNSKI (1977)).
*G2* $\Rightarrow$ *LT*

*Proof.* Let $\Sigma$ be any formal system adequate for recursive number theory.

(1) $\Sigma \nvdash \Psi$
(2) $\Sigma + \sim \Psi$ :consistent
(3) $\Sigma + \sim \Psi \nvdash \text{con}(\Sigma + \sim \Psi)$ ; G2
(4) $\Sigma + \sim \Psi \nvdash \sim Pr(\ulcorner \sim \Psi \rightarrow \overline{0} = \overline{1} \urcorner)$
(5) $\Sigma + \sim \Psi \nvdash \sim Pr(\ulcorner \Psi \urcorner)$
(6) $\Sigma \nvdash (\sim \Psi \rightarrow \sim Pr(\ulcorner \Psi \urcorner))$ ; deduction theorem
(7) $\Sigma \nvdash (Pr(\ulcorner \Psi \urcorner) \rightarrow \Psi)$

THEOREM 4.
*G2 + Faithfulness* $\Rightarrow$ *G1 (Gödel's First Incompleteness Theorem)*

*Proof.*

(1) $(\Sigma \nvdash \text{Con}) \& \forall \Psi (\Sigma \vdash \Psi \; or \; \Sigma \vdash \sim \Psi)$ ; G2 & not G1
(2) $\Sigma \vdash Pr(\ulcorner \bot \urcorner)$ ; (1)
(3) $\Sigma \vdash \bot$ ; Faithfulness
(4) $\Sigma \vdash \text{Con}$ ; $\vdash \bot \Rightarrow \vdash A$, for any $A$

THEOREM 5.
$LT \Leftrightarrow \forall i = 0,1 \; \forall j = 0,1 (F_i : \text{a formula} \; \& \vdash (Pr(\ulcorner F_i \urcorner) \rightarrow F_j)$
$\& \vdash (Pr(\ulcorner F_j \urcorner) \rightarrow F_i) \quad \Rightarrow \quad \vdash F_i)$

*Proof.*
$(\Rightarrow)$

(1) $\vdash Pr(\ulcorner F_i \urcorner) \rightarrow F_j$
(2) $\vdash Pr(\ulcorner F_j \urcorner) \rightarrow F_i$
(3) $\vdash Pr(\ulcorner F_i \wedge F_j \urcorner) \leftrightarrow Pr(\ulcorner F_i \urcorner) \wedge Pr(\ulcorner F_j \urcorner)$
(4) $\vdash Pr(\ulcorner F_i \wedge F_j \urcorner) \rightarrow F_i \wedge F_j$ ; (1), (2), (3)
(5) $\vdash F_i \wedge F_j$ ; LT
(6) $\vdash F_i \; \& \vdash F_j$ (for all $i, j = 0, 1$)

$(\Leftarrow)$ Special case when for all $i \& j$, $F_i = F_j$

LEMMA 6.
$\forall G(\vdash G \leftrightarrow \sim Pr(\ulcorner G \urcorner) \Rightarrow \vdash Con \leftrightarrow G)$

*Proof.*

(1) $\vdash \bot \rightarrow G$, for all $G$
(2) $\vdash (Pr(\ulcorner \bot \urcorner)) \rightarrow Pr(\ulcorner G \urcorner)$; Adequacy and Conditionalization
(3) $\vdash (\sim Pr(\ulcorner G \urcorner) \rightarrow \text{Con})$
(4) $\vdash (G \rightarrow \text{Con})$
(5) $\vdash (Pr(\ulcorner G \urcorner)) \rightarrow \sim G$ ; Definition of G
(6) $\vdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner \sim G \urcorner)$ ; Adequacy and Conditionalization
(7) $\vdash Pr(\ulcorner G \urcorner) \rightarrow Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner)$ ; Formal Adequacy
(8) $\vdash Pr(\ulcorner G \urcorner) \rightarrow Pr(\ulcorner \sim G \urcorner)$ ; (6), (7)
(9) $\vdash Pr(\ulcorner G \urcorner) \rightarrow \sim \text{Con}$ ; $\vdash \text{Con} \rightarrow \sim [Pr(\ulcorner G \urcorner) \wedge Pr(\ulcorner \sim G \urcorner)]$
(10) $\vdash \text{Con} \rightarrow G$

REMARK 7.

(i) *Any sentence asserting its own unprovability in the theory is a consistency sentence. The Gödel sentence is of this form.*
(ii) *Since its existential generalization is a consistency sentence, it at least entails a consistency sentence; and vice versa.*
(iii) *Any two Gödel sentences are provably equivalent, just like any two Henkin's sentences.*

For all the statements below, let $G$ be the Gödel sentence and let $Pr(x)$ satisfy the three conditions of a standard provability predicate and Faithfulness condition. The resulting axiomatic theory in question will be denoted by $\Sigma$.

THEOREM 8.
$LT \Rightarrow (\omega - con(\Sigma) \Rightarrow \Sigma \nvdash (Con \rightarrow \sim Pr(\ulcorner \sim G \urcorner)))$

*Proof.*

(1) $\vdash \text{Con} \rightarrow \sim Pr(\ulcorner \sim G \urcorner)$ ; assumption
(2) $\vdash G \rightarrow \sim Pr(\ulcorner \sim G \urcorner)$ ; Lemma 6
(3) $\vdash Pr(\ulcorner \sim G \urcorner) \rightarrow \sim G$ ; contrapositive
(4) $\vdash \sim G$ ; LT
(5) $\Sigma : \omega -$ inconsistent ; G1 by Theorem 4 (using Faithfulness)

THEOREM 9.
$[\omega - con(\Sigma) \Rightarrow \Sigma \nvdash Con \rightarrow \sim Pr(\ulcorner \sim G \urcorner)] \Rightarrow [\omega - con(\Sigma) \Rightarrow \Sigma \nvdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)]$

*Proof.*

(1) $\omega - \text{con}(\Sigma)$ ; assumption
(2) $\Sigma + \{\text{Con}\} \nvdash \sim Pr(\ulcorner \sim G \urcorner)$ ; deduction theorem, hypothesis
(3) $(Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)) \Leftrightarrow (\sim Pr(\ulcorner \sim G \urcorner) \vee Pr(\ulcorner G \urcorner)) \Leftrightarrow \sim Pr(\ulcorner \sim G \urcorner) \Leftrightarrow \sim G$ ; Definition of G
(4) $\Sigma + \{\text{Con}\} \nvdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)$ ; (2), (3)
(5) $\Sigma \nvdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)$ ; (4)

Note that the hypothesis $\omega - \text{con}(\Sigma)$ of the right-hand sentence in Theorem 8 and Theorem 9 can be weakened to $\text{con}(\Sigma)$.

THEOREM 10.
$(\omega - con(\Sigma) \Rightarrow \Sigma \nvdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)) \Rightarrow G1$

*Proof.*
Straightforward from the rightmost $\Leftrightarrow$ in proof step (3) of Theorem 9.

Thus we have established:

COROLLARY 11.
*The following are equivalent in $\Sigma$ :*

1. $LT$
2. $\forall i = 0,1, \forall j = 0,1$ $(F_i$ : *formula* & $(\vdash Pr(\ulcorner F_i \urcorner) \rightarrow F_j)$ & $(\vdash Pr(\ulcorner F_j \urcorner) \rightarrow F_i) \Rightarrow \vdash F_i)$
3. $\omega - con(\Sigma) \Rightarrow \Sigma \nvdash Con \rightarrow \sim Pr(\ulcorner \sim G \urcorner)$
4. $\omega - con(\Sigma) \Rightarrow \Sigma \nvdash Pr(\ulcorner Pr(\ulcorner G \urcorner) \urcorner) \rightarrow Pr(\ulcorner G \urcorner)$
5. $G1$
6. $G2$

*Proof.* Theorem 5, Theorem 8 (using Faithfulness indirectly), Theorem 9, Theorem 10, and Fact 3. G2 is a consequence of G1 under our assumption stated right before Theorem 8.

REMARK 12.

(i) It is to note the provable equivalence between G1 and G2 in the underlying theory in question, even if G2 differs from G1 in the subtlety of what it says. For much depends on what we take to be a formal statement of the consistency of the system, e.g., requiring that $Pr(x)$ be a standard provability predicate or not (e.g., lacking Formal Adequacy) and that the formal consistency statement Con satisfy the condition : for all sentences $A$, $\vdash \text{Con} \rightarrow \sim [Pr(\ulcorner A \urcorner) \wedge Pr(\ulcorner \sim A \urcorner)]$.

(ii) If we do not assume $Pr(x)$ to be $\Sigma_1$ then Corollary 11 does not hold, for e.g., for Feferman's $\Pi_1$ binumeration of axioms of Peano Arithmetic the resulting $Pr(x)$ satisfies our conditions for a standard proof predicate, the corresponding G1 is true (both $G$ and $\sim G$ is unprovable) but G2 is false (Con is provable).

## References


1. G. Kreisel, *About Logic and Logicians* II, : Mathematics, A palimpsest of essays by Georg Kreisel, selected and arranged by Piergiorgio Odifreddi, Manuscript, 1993.
2. G. Kreisel, A./Lévy, *Reflection principles and their use for establishing the complexity of axiomatic systems*, Z. Math. Logik Grundlag. Math. **14** (1968), 97–142.
3. S. Kripke, *A talk at UCLA Set Theory Meeting* (1967).
4. M. H. Löb, *Solution of a problem by Leon Henkin*, [Proc. ICM 1954] (1954), 405–406.
5. ———, *Solution of a problem by Leon Henkin*, J. Symbolic Logic **20** (1955), 115–118.
6. C. Smorynski, *The incompleteness theorems*, Handbook of mathematical logic, Amsterdam-New York-Oxford(NHP), 1977.
7. ———, *The development of self-reference. Löb's theorem*, manuscript (1991).
8. R. M. Solovay, *Provability interpretations of modal logic*, Israel J. Math. **25** (1976), 287–304.
9. ———, *Explicit Henkin sentences*, J. Symbolic Logic **50** (1985), 91–93.


Department of Mathematics
Yonsei University
Seoul 120-749, Korea