J. Korean Math. Soc. **31** (1994), No. 4, pp. 669–678

EQUI AND UNIFORM STABILITY IN DYNAMICAL POLYSYSTEMS

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1. Introduction and Preliminaries

The subject of Liapunov functions constitutes a central theme in the theory of differential equations. It provides powerful tools that can be used to study the behavior of the solutions.

The classical theorem of Liapunov on stability of the zero solutions for a given differential equation makes use of an auxiliary function which has to be positive definite. Also, the time derivative of this function, as computed along the solution, has to be negative definite. This auxiliary function is called a Liapunov function in the theory of differential equations or more generally dynamical systems theory.

Kalouptsidis, Tsinias and Bacciotti studied stability concepts which can be described by suitable versions of Liapunov functions in the theory of control systems and the theory of dynamical polysystems.

Kalouptsidis[3, Theorem 5.7] proved the equivalence of absolute asymptotic stability (A.A.S.) and the existence of a Liapunov function in the topological abstraction (X, D) of the control system, where D is a collection of dynamical systems on a metric space (X, d). And Tsinias proved that if a compact subset of the state space X is A.A.S., then it has a Liapunov function defined on the interior of the region of attraction [4, Theorem 18]. They showed that Liapunov functions guarantee A.A.S. in dynamical polysystems [5, Theorem 24].

Also, Tsinias and Kalouptsidis [6, Theorem 5] generalized a stability theorem in a single dynamical system [2, Theorem V. 4.5], i.e., α-stability for a closed subset M in a locally compact space X.

In this paper we analyze the stability concepts, mainly equistability and uniform stability, for a closed subset in dynamical polysystems by

Received July 30, 1993. Revised November 2, 1993.

The present study was supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1992, Project No. BSRI-92-110.

means of suitably defined Liapunov functions, which generalize theorems presented in [2, Theorem V. 4.7 and Theorem V. 4.9] and give a counterexample for an equistability theorem of Bhatia and Szegö [2, Theorem V. 4.7].

We recall some definitions from [5].

Let X be a locally compact metric space with a metric d. A *dynamical system* on X is a continuous map $\pi : \mathbb{R} \times X \to X$ satisfying the conditions

(i) $\pi(0, x) = x$ for all $x \in X$,
(ii) $\pi(t, \pi(s, x)) = \pi(t + s, x)$ for all $x \in X$ and $t, s \in \mathbb{R}$.

We write $\pi(t, x)$ simply as tx. In line with this notation, if $A \subset \mathbb{R}$ and $M \subset X$, then AM is the set $\{tx : t \in A, x \in M\}$. If $A = \{t\}$ or $M = \{x\}$, then we simply write tM and Ax for $\{t\}M$ and $A\{x\}$, respectively. For any $x \in X$, the set $\mathbb{R}x$ is called the *trajectory* through x.

We call a family of dynamical systems $\{\pi_i : i \in I\}$ a *dynamical polysystem* on X. Dynamical polysystems arise in control theory.

If 2^X denotes the set of all subsets of X and $\mathbb{R}^+$ the set of nonnegative real numbers $[0, +\infty)$, the *reachable map* of the polysystem $\{\pi_i : i \in I\}$ is the multivalued map $R : \mathbb{R}^+ \times X \to 2^X$ defined by

$$\begin{aligned} R(t, x) = \Big\{ & y \in X : \text{there exist an integer } k, i_1, \cdots, i_k \text{ in } I \\ & \text{and } t_1, \cdots, t_k \text{ in } \mathbb{R}^+ \text{ such that } \sum_{i=1}^{n} t_i = t \text{ and} \\ & y = \pi_{i_k}(t_k, \pi_{i_{k-1}}(t_{k-1}, \cdots, \pi_{i_1}(t_1, x)), \cdots)) \Big\}. \end{aligned}$$

Also, we define $R(x) = \bigcup_{0 \le t < \infty} R(t, x)$ for all $x \in X$. It should be noted that while $R(t, x)$ is assumed continuous, $R(x)$ is not necessarily so.

The stability theory of polysystems is roughly concerned with how the reachable sets $R(t, x)$ are put together on the state space X.

For a point x in X, a subset M in X and $\varepsilon > 0$, we denote

$$\begin{aligned} d(M, x) &= \inf\{d(x, y) : y \in M\}, \\ B(M, \varepsilon) &= \{x \in X : d(M, x) < \varepsilon\}. \end{aligned}$$

Also, we denote by $\overline{M}$ the closure of the set M.

We say that a closed subset M of X is

stable if for each x in M and $\varepsilon > 0$, there exists a $\delta > 0$ such that $R(B(x,\delta)) \subset B(M,\varepsilon)$,

equistable if for each $x \notin M$, there exists an $\varepsilon > 0$ such that $x \notin \overline{R(B(M,\varepsilon))}$;

uniformly stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $R(B(M,\delta)) \subset B(M,\varepsilon)$.

Finally, a multivalued map $\Gamma : X \to 2^X$ is called a *c.c. map* if for each compact set K in X and each x in K with $\Gamma(x) \not\subset K, \Gamma(x) \cap \partial K \neq \emptyset$, where ∂K is the boundary of K.

It is not hard to show that the reachable map R is a c.c. map.

2. Equistability and Uniform Stability

Bhatia and Szegö [2] developed several notions of stability in a single dynamical system and provided criteria for these in terms of Liapunov functions. First, we investigate some stability behaviors which generalize Propositions V. 4.2 and V. 4.4 in [2].

PROPOSITION 1. *If a closed subset M of X is uniformly stable, then it is both stable and equistable.*

Proof. It is clear that M is stable. For each $x \notin M$, let $\varepsilon = d(M,x) > 0$. Then we have $R(B(M,\delta)) \subset B(M,\varepsilon/2)$ for some $\delta > 0$. We claim that $B(x,\varepsilon/2) \cap R(B(M,\delta)) = \emptyset$. Assuming the contrary, there exists an element y in $B(x,\varepsilon/2) \cap R(B(M,\delta))$. Since $y \in R(B(M,\delta)) \subset B(M,\varepsilon/2)$, we have $D(M,y) < \varepsilon/2$. This implies that

$$\varepsilon = d(M,x) \leq d(M,y) + d(y,x) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

a contradiction. Thus we must have $x \notin \overline{R(B(M,\delta))}$. It follows that M is equistable.

PROPOSITION 2. *Let M be a compact subset of X. If M is stable, then it is uniformly stable.*

Proof. Let $\varepsilon > 0$ be given. Since M is stable, for each $x \in M$ there is a $\delta_x > 0$ such that $R(B(x,\delta_x)) \subset B(M,\varepsilon)$. Also, there is a $\delta > 0$

such that $B(M,\delta) \subset \bigcup_{x\in M} B(x,\delta_x)$ by compactness of M. Thus

$$R(B(M,\delta)) \subset R(\bigcup_{x\in M} B(x,\delta_x)) = \bigcup_{x\in M} R(B(x,\delta_x)) \subset B(M,\varepsilon).$$

This means that M is uniformly stable.

PROPOSITION 3. *If a compact subset M of X is equistable, then M is uniformly stable.*

Proof. Suppose that M is not uniformly stable. Then there exists an $\varepsilon > 0$ such that for each $\delta > 0, R(B(M,\delta)) \not\subset B(M,\varepsilon)$. We can choose an $\alpha > 0$ such that $\overline{B(M,\alpha)}$ is compact and $\overline{B(M,\alpha)} \subset B(M,\varepsilon)$. It is clear that $R(B(M,\alpha/n) \not\subset \overline{B(M,\alpha)}$ for each positive integer n. It follows that $R(x_n) \not\subset \overline{B(M,\alpha)}$ for some sequence (x_n) in $B(M,\alpha/n) \subset \overline{B(M,\alpha)}$. Moreover, we have $R(x_n) \cap \partial B(M,\alpha) \neq \emptyset$ because R is a c.c. map. Thus we can choose a sequence (y_n) in $R(x_n) \cap \partial B(M,\alpha)$. This sequence (y_n) has a subsequence which converges to some point y in $\partial B(M,\alpha)$ since $\partial B(M,\alpha)$ is compact. But $y \notin \overline{R(B(M,\beta))}$ for some $\beta > 0$ by the equistability of M. Clearly, the set $X - \overline{R(B(M,\beta))}$ is a neighborhood of y. We can select an integer m such that $y_m \notin \overline{R(B(M,\beta))}$ with $\alpha/m < \beta$. However we have $y_m \in R(x_m) \subset R(B(M,\alpha/m)) \subset R(B(M,\beta))$. This is a contradiction. Therefore M is uniformly stable.

For a closed subset M of X, we have the following.

PROPOSITION 4. *If a closed subset M of X is either stable or equistable, then it is positively invariant.*

Proof. Assume that M is stable but not positively invariant. Then $R(x) \not\subset M$ for some x on M. Taking $y \in R(x) - M$, we can choose an $\varepsilon > 0$ such that $y \notin B(M,\varepsilon)$. By the stability of M we have $R(B(x,\delta)) \subset B(M,\varepsilon)$ for some $\delta > 0$. Thus $y \in R(x) \subset R(B(x,\delta)) \subset B(M,\varepsilon)$, a contradiction. Now, assume that M is equistable but not positively invariant. Then we can select a point y in $R(x) - M$ for some point x on M. By the equistability of $M, y \notin \overline{R(B(M,\varepsilon))}$ for some $\varepsilon > 0$. Therefore

$$y \in R(x) \subset R(M) \subset R(B(M,\varepsilon)) \subset \overline{R(B(M,\varepsilon))}.$$

This contradicts the fact that $y \notin \overline{R(B(M,\varepsilon))}$. Consequently, M must be positively invariant. This completes the proof.

REMARK 1. Propositions 2 and 3 are a generalization of Proposition V. 4.2 in [2] and its converse is generalized as Proposition 1. Also, Proposition 4 generalizes Proposition V. 4.4 in [2].

REMARK 2. For the differential equation on $\mathbb{R}$

$$x' = x(\sin(\pi/x))^2$$

existence and uniqueness of solutions are true, and hence all continuity requirements of $R(t,x)$, for example, are good. Since all points $x = \frac{1}{n}$, with n positive integer, are rest points, the origin $x = 0$ is stable. Nevertheless, any Liapunov function $\phi(x)$ with $\phi(0) = 0$ and $\phi(x) > 0$ otherwise must be nonincreasing in $(\frac{1}{n+1}, \frac{1}{n})$. The only way this is possible is by having discontinuities at $x = \frac{1}{n}$. Therefore we can not generalize the result about stability for a single dynamical system, which is appeared in [2, Theorem V. 4.5].

In [2] the notion of equistability for a closed set in a single dynamical system was introduced and the following result presented.

THEOREM (Bhatia-Szegö)[2, Theorem V. 4.7]. *A closed subset M of X is equistable if and only if there is a function $\phi : X \to \mathbb{R}^+$ such that*

(1) *$\phi(x) = 0$ for $x \in M$ and $\phi(x) > 0$ for $x \notin M$,*
(2) *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) \leq \varepsilon$ if $d(M,x) \leq \delta$,*
(3) *$\phi(tx) \leq \phi(x)$ for $x \in X$ and $t \geq 0$.*

However, the following example shows that this theorem does not hold in general.

EXAMPLE 5. Let $X = \{(x,y) \in \mathbb{R}^2 : x \leq 0 \leq y\} \cup \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 1\}$. Define a function $\pi : \mathbb{R} \times X \to X$ by

$$\pi(t,(x,y)) = \begin{cases} (x, ye^{xt}) & \text{if } x \leq 0, \\ (x, (y-1)e^{-xt} + 1) & \text{if } x > 0, \end{cases}$$

Then π satisfies the conditions of dynamical system on X. We consider a closed subset $M = \{(x,y) \in X : x \geq 0, y = 1\}$ and define a function $\phi : X \to \mathbb{R}^+$ by

$$\phi(x,y) = \begin{cases} -x & \text{if } x < 0, \\ |y-1| & \text{if } x \geq 0. \end{cases}$$

Then ϕ satisfies the following conditions :

(1) $\phi(x,y) = 0$ if and only if $(x,y) \in M$,

(2) For any $\varepsilon > 0, \phi(x,y) < \varepsilon$ whenever $d(M,(x,y)) < \varepsilon$,

(3) For each $(x,y) \in X$ and $t \in \mathbb{R}^+, \phi(t(x,y)) \leq \phi(x,y)$.

Note that $(0,0) \notin M$ but $(0,0) \in \overline{\mathbb{R}^+B(M,\varepsilon)}$ for any $\varepsilon > 0$. This shows that M is not equistable.

Now we improve the above theorem for equistability.

THEOREM 6. *A closed subset $M \subset X$ is equistable if and only if there is a function $\phi : X \to \mathbb{R}^+$ satisfying*

(1) *$\phi(x) = 0$ if and only if $x \in M$,*

(2) *For any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) < \varepsilon$ if $d(M,x) < \delta$,*

(3) *For all $x \in X$ and $t \in \mathbb{R}^+, \phi(tx) \leq \phi(x)$,*

(4) *For any $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $\phi(x) - \varepsilon < \phi(y)$.*

Proof. By the equistability of M, we can take a set $I(x) = \{\delta > 0 : x \notin \overline{\mathbb{R}^+B(M,\delta)}\}$. Define a function $\phi : X \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} \sup I(x) & \text{if } x \notin M, \\ 0 & \text{if } x \in M. \end{cases}$$

It is evident that $\phi(x) \leq d(M,x)$ for every $x \in X$.

Let $x \notin M$. Then $\phi(x) \geq \delta > 0$ for some $\delta \in I(x)$. Thus $\phi(x) = 0$ implies $x \in M$ and by the definition of ϕ the reverse is obvious. Therefore ϕ satisfies the condition (1). For the condition (2), we note that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) \leq d(M,x) < \delta$ if $d(M,x) < \delta$. In order to show that ϕ satisfies (3), let $x \in M$. Then $tx \in M$ by Proposition 4. Thus $\phi(tx) = 0 = \phi(x)$. Let $x \notin M$ and suppose that there is an $r \in I(tx) - I(x)$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that $tB(x,\delta) \subset B(tx,\varepsilon)$. Since $x \in \overline{\mathbb{R}^+B(M,r)}, B(x,\delta) \cap \mathbb{R}^+B(M,r) \neq \emptyset$. Then we have $sy \in B(x,\delta)$ for some $y \in B(M,r)$ and $s \in \mathbb{R}^+$. Therefore

$$(s+t)y = t(sy) \in tB(x,\delta) \subset B(tx,\varepsilon).$$

Since $(s+t)y \in \mathbb{R}^+B(M,r), B(tx,\varepsilon) \cap \mathbb{R}^+B(M,r) \neq \emptyset$. This implies that $tx \in \overline{\mathbb{R}^+B(M,r)}$, a contradiction. Consequently, $I(tx) \subset I(x)$ and

it follows that $\phi(tx) \leq \phi(x)$. Finally, we show that ϕ satisfies (4). For any $x \notin M$ and $\varepsilon > 0$, there exists a $\delta > 0$ in $I(x)$ such that $\phi(x) - \varepsilon < \delta$. Moreover there is an $\alpha > 0$ such that $B(x,\alpha) \cap \mathbb{R}^+ B(M,\delta) = \emptyset$ because $x \notin \overline{\mathbb{R}^+ B(M,\delta)}$. Then we have $\phi(y) \geq \delta > \phi(x) - \varepsilon$ since $\delta \in I(y)$ for all $y \in B(x,\alpha)$.

Conversely, we show that for each $x \notin M$ there is a $\delta > 0$ such that $x \notin \overline{\mathbb{R}^+ B(M,\delta)}$. Let $x \notin M$ and $\phi(x) = \varepsilon > 0$. By the condition (2), there is a $\delta > 0$ such that $\phi(y) < \varepsilon/2$ whenever $y \in B(M,\delta)$. Suppose $x \in \overline{\mathbb{R}^+ B(M,\delta)}$. Then we have $B(x,1/n) \cap \mathbb{R}^+ B(M,\delta) \neq \emptyset$ for each positive integer n. Thus $t_n x_n \in B(x,1/n)$ for some points $x_n \in B(M,\delta)$ and $t_n \in \mathbb{R}^+$. Also there is an $\alpha > 0$ such that $\phi(y) > \varepsilon/2$ if $d(x,y) < \alpha$ by the condition (4). We can take a positive integer m with the property that $t_m x_m \in B(x,\alpha)$ since $t_n x_n \to x$ as $n \to \infty$. Then $\varepsilon/2 < \phi(t_m x_m) < \phi(x_m) < \varepsilon/2$, a contradiction.

REMARK. The function ϕ in the above theorem need not be continuous [2, Example V. 4.11].

Now, we have the following equistability theorem in dynamical polysystems, which is a result parallel to Theorem 6, in a single dynamical system.

THEOREM 7. *A closed subset M of X is equistable if and only if there exists a function $\phi : X \to \mathbb{R}^+$ such that*

(1) *$\phi(x) = 0$ if and only if $x \in M$,*
(2) *For any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) < \varepsilon$ whenever $d(M,x) < \delta$,*
(3) *$\phi(y) \leq \phi(x)$ for each $x \in X$ and $y \in R(x)$,*
(4) *For every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $\phi(x) - \varepsilon < \phi(y)$ whenever $d(x,y) < \delta$.*

Proof. We replace $\mathbb{R}^+ B(M,\cdot)$ by $R(B(M,\cdot))$ about the condition of the set $I(x)$ in the proof of Theorem 6. We only show that $I(y) \subset I(x)$ with $y \in R(t,x)$ for the condition (3). Assuming the contrary, there is a $\delta \in I(y) - I(x)$. Since $y \notin \overline{R(B(M,\delta))}$, we have $B(y,\alpha) \cap R(B(M,\delta)) = \emptyset$ for some $\alpha > 0$. Let $y \in R(t,x)$. Then there are $i_1, \cdots, i_n \in I, t_1, \cdots, t_n \in \mathbb{R}^+$ and $x_1, \cdots, x_{n-1} \in X$ such that

$$\sum_{i=1}^{n} t_i = t$$

and

$$x_1 = \pi_{i_1}(t_1, x), x_2 = \pi_{i_2}(t_2, x_1), \cdots, y = \pi_{i_n}(t_n, x_{n-1}).$$

Also, there are $\alpha_0, \cdots, \alpha_{n-1} > 0$ such that

$$\pi_{i_1}(t_1, B(x, \alpha_0)) \subset B(x_1, \alpha_1), \cdots, \pi_{i_n}(t_n, B(x_{n-1}, \alpha_{n-1})) \subset B(y, \alpha)$$

by the continuity of $\pi_{i_k}, k = 1, \cdots, n$. It is clear that $x \in \overline{R(B(M, \delta))}$ since $\delta \notin I(x)$. This implies that $B(x, \alpha_0) \cap R(B(M, \delta))$ contains a point, say z. Taking

$$w = \pi_{i_n}(t_n, \cdots, \pi_{i_2}(t_2, \pi_{i_1}(t_1, z)), \cdots),$$

we have $w \in B(y, \alpha)$. Also, we have

$$w \in R(t, z) \subset R(z) \subset R(R(B(M, \delta)) = R(B(M, \delta)).$$

It follows that $B(y, \alpha) \cap R(B(M, \delta)) \neq \emptyset$, which is a contradiction. Hence we have $\phi(y) = \sup I(y) \leq \sup I(x) = \phi(x)$. The proof that ϕ satisfies the remaining conditions (1), (2) and (4) is omitted because it is parallel to the proof of Theorem 6.

For the converse, we can proceed as in the proof of Theorem 6 replacing $\mathbb{R}^+ B(M, \cdot)$ by $R(B(M, \cdot))$.

Finally, we obtain the following uniform stability theorem.

THEOREM 8. *A closed subset M of X is uniformly stable if and only if there exists a function $\phi : X \to \mathbb{R}^+$ such that*

(1) *For any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) \geq \delta$ whenever $d(M, x) \geq \varepsilon$,*
(2) *For any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) < \varepsilon$ whenever $d(M, x) < \delta$,*
(3) *$\phi(y) \leq \phi(x)$ for any $x \in X$ and $y \in R(x)$.*

Proof. Define a function $\phi : X \to \mathbb{R}^+$ by

$$\phi(x) = \sup_{t \in \mathbb{R}^+} \frac{\ell(t, x)}{1 + \ell(t, x)},$$

where $\ell(t,x) = \sup\{d(M,y) : y \in R(t,x)\}$. We show that ϕ satisfies (1), (2) and (3). For any $\varepsilon > 0$, put $\delta = \varepsilon/(1+\varepsilon) > 0$. Note that $d(M,x) = \ell(0,x)$. If $d(M,x) \geq \varepsilon$, then

$$\phi(x) > \frac{\ell(0,x)}{1+\ell(0,x)} > \frac{\varepsilon}{1+\varepsilon} = \delta.$$

This implies that ϕ satisfies (1).

For the condition (2), we note that for each $\varepsilon > 0, \alpha/(1+\alpha) < \varepsilon$ for some $\alpha > 0$. Since M is uniformly stable, there is a $\delta > 0$ such that $R(B(M,\delta)) \subset B(M,\alpha)$. For any $t \in \mathbb{R}^+$, we have

$$R(t,x) \subset R(x) \subset R(B(M,\delta)) \subset B(M,\alpha),$$

if $d(M,x) < \delta$. It follows that $\ell(t,x) \leq \alpha$. Therefore $\phi(x) \leq \alpha/(1+\alpha) < \varepsilon$.

Now, to show that ϕ satisfies (3) we let $y \in R(t,x)$. Then

$$\begin{aligned}\ell(s,y) &= \sup_{z\in R(s,y)} d(M,z) \leq \sup_{z\in R(s,R(t,x))} d(M,z)\\ &= \sup_{z\in R(t+s,x)} d(M,z) = \ell(t+s,x).\end{aligned}$$

Therefore ϕ satisfies (3) from the fact that

$$\begin{aligned}\phi(y) &= \sup_{s\in\mathbb{R}^+}\frac{\ell(s,y)}{1+\ell(s,y)} \leq \sup_{s\in\mathbb{R}^+}\frac{\ell(t+s,x)}{1+\ell(t+s,x)}\\ &= \sup_{s\geq t}\frac{\ell(s,x)}{1+\ell(s,x)} \leq \sup_{s\in\mathbb{R}^+}\frac{\ell(s,x)}{1+\ell(s,x)} = \phi(x).\end{aligned}$$

Conversely, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\phi(x) \geq \delta$ if $d(M,x) \geq \varepsilon$ by the assumption. Also, there is an $\alpha > 0$ such that $\phi(x) < \delta$ if $d(M,x) < \alpha$. Let $x \in R(B(M,\alpha))$. Then $x \in R(y)$ for some $y \in B(M,\alpha)$. If we assume that $x \notin B(M,\varepsilon)$, then we have $d(M,x) \geq \varepsilon$ and so $\phi(x) \geq \delta$. Since $y \in B(M,\alpha), d(M,y) < \alpha$ and thus $\phi(y) < \delta$. Also we have $\phi(x) \leq \phi(y)$. Hence $\delta \leq \phi(x) \leq \phi(y) < \delta$, a contradiction. Therefore $x \in B(M,\varepsilon)$. This means that M is uniformly stable and the proof is completed.

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