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A POINTED BLASCHKE MANIFOLD IN EUCLIDEAN SPACE

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1. Introduction

Submanifolds of Euclidean spaces have been studied by examining geodesics of the submanifolds viewed as curves of the ambient Euclidean spaces ([3], [7], [8], [9]). K.Sakamoto ([7]) studied submanifolds of Euclidean space whose geodesics are plane curves, which were called submanifolds with planar geodesics. And he completely classified such submanifolds as either Blaschke manifolds or totally geodesic submanifolds. We now ask the following: If there is a point p of the given submanifold in Euclidean space such that every geodesic of the submanifold passing through p is a plane curve, how much can we say about the submanifold ?

In the present paper, we study submanifolds of Euclidean space with such property.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Euclidean m-space E^m by an immersion x. Then the metric tensor on M is naturally induced from that of E^m. We use the same notation $\langle , \rangle$ for the metrics unless stated otherwise. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on M and E^m respectively. Then, we have the so-called Gauss equation $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$, where X and Y denote vector fields on M and h is the second fundamental form.

The equation of Weingarten is given by $\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$, where A_ξ is the Weingarten map associated with a normal vector field ξ to

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M and $\nabla^{\perp}$ the normal connection in the normal bundle $T^{\perp}M$. As is well known, the Weingarten map A_{ξ} and the second fundamental form h are related by $< A_{\xi}X, Y >=< h(X,Y), \xi >$ for all vector fields X and Y on M and ξ normal to M. We now define the van der Waerden-Bortolotti covariant derivative of h as

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for all vector fields X, Y and Z on M. We denote $(\bar{\nabla}_X h)(Y,Z)$ by $(\bar{\nabla}h)(X,Y,Z)$ which is a tensor field of type (1,3). Let R be the curvature tensor of M. The Gauss equation is given by

$$\begin{aligned} &< R(X,Y)Z,W > \\ &=< h(X,W),h(Y,Z) > - < h(Y,W),h(X,Z) > . \end{aligned} \tag{1.1}$$

We also obtain the Codazzi equation

$$(\bar{\nabla}h)(X,Y,Z) - (\bar{\nabla}h)(Y,X,Z) = 0. \tag{1.2}$$

The submanifold M in Euclidean space E^m is said to be *isotropic* at $p \in M$ if the normal curvature of curves passing through p is independent of the choice of the curve, that is, $< h(t,t), h(t,t) >$ does not depend on the choice of the unit vector t tangent to M at p. By B. O'Neill [8], M is isotropic at p if and only if $< h(t,t), h(t,t^{\perp}) >= 0$ for all unit vectors t and $t^{\perp}$ perpendicular to t.

For a point $p \in M$ and a unit vector t tangent to M at p, the vector t and the normal space $T_p^{\perp}M$ of M at p form an $(m-n+1)$-dimensional affine space $E(p;t)$ in E^m through p. The intersection of M with $E(p;t)$ gives rise to a curve in a neighborhood of p which is called the *normal section* at p in the direction t (See [3], [5]).

Let M be a complete Riemannian manifold and let p and q be points of M. The *link* $\Lambda(p,q)$ from p to q is defined as

$$\Lambda(p,q) = \{\gamma'(q) \in U_qM | \gamma \in Seg(p,q)\},$$

where γ is assumed to be parametrized by one length, $Seg(p,q)$ denotes the set of minimal geodesics joining p to q and U_qM is the unit tangent space of M at q. A compact Riemannian manifold M is called a *Blaschke manifold at* p if for every q in $Cut(p)$ the link $\Lambda(p,q)$ is a great sphere of U_qM, where $Cut(p)$ denotes the cut locus of p. The manifold M is said to be a *Blaschke manifold* if it is a Blaschke manifold at every point of M.

3. Submanifolds of Euclidean space with planar geodesics through a point

Let M be an n-dimensional submanifold of an m-dimensional Euclidean space E^m by an isometric immersion $x : M \longrightarrow E^m$.

We prove

LEMMA 1. *Let M be an n-dimensional submanifold of a Euclidean space E^m. If γ is a planar geodesic of M, then γ is a normal section of M along γ.*

Proof. We may assume that γ is parametrized by the arc length s. Let $\gamma'(s) = T$. Then, $\gamma''(s) = h(T,T)$ and $\gamma'''(s) = -A_{h(T,T)}T + (\bar{\nabla}h)(T,T,T)$. Since γ is planar, $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = 0$ along γ. Thus, we have

$$T \wedge A_{h(T,T)}T \wedge h(T,T) - T \wedge h(T,T) \wedge (\bar{\nabla}h)(T,T,T) = 0. \qquad (3.1)$$

Suppose $T \wedge A_{h(T,T)}T \neq 0$ at $\gamma(s_0)$ for some $s_0 \in \mathrm{Dom}\gamma$ where $\mathrm{Dom}\gamma$ denotes the domain of γ. Then, there exists an open interval I such that $s_0 \in I \subset \mathrm{Dom}\gamma$ and $T \wedge A_{h(T,T)}T \neq 0$ at $\gamma(s)$ for all $s \in I$. We can choose vector fields $X_3, X_4, \cdots X_n$ tangent to M for every $s \in I$. Taking the exterior product of $A_{h(T,T)}T \wedge X_3 \wedge \cdots \wedge X_n$ with (3.1), we obtain

$$T \wedge A_{h(T,T)}T \wedge X_3 \wedge \cdots \wedge X_n \wedge h(T,T) \wedge (\bar{\nabla}h)(T,T,T) = 0,$$

from which,

$$h(T,T) \wedge (\bar{\nabla}h)(T,T,T) = 0$$

at $\gamma(s)$ for every $s \in I$. Therefore, (3.1) gives

$$T \wedge A_{h(T,T)}T \wedge h(T,T) = 0.$$

Since $T \wedge A_{h(T,T)}T \neq 0$ for $s \in I$, $h(T,T) = 0$ at $\gamma(s)$ for every $s \in I$, which is a contradiction. Consequently, we have

$$T \wedge A_{h(T,T)}T = 0 \text{ and } h(T,T) \wedge (\bar{\nabla}h)(T,T,T) = 0 \qquad (3.2)$$

along γ. The curve γ lies in $\gamma(s) + Sp\{T(s), h(T(s),T(s))\}$ for each fixed s. The uniqueness theorem of geodesic implies that γ is a normal section of M in the direction $\gamma'(s)$ along γ. □

We now define

(*) : There is a point o in M such that every geodesic through o is planar.

We now suppose that M admits (*). Without loss of generality we may assume that the point o is the origin of E^m. Let γ be a geodesic of M passing through o and let γ be parametrized by the are length s. Let $\gamma(0) = o$. As in Lemma 3.1, we have

$$\gamma'(s) = T,\ \gamma''(s) = h(T,T),\ \gamma'''(s) = -A_{h(T,T)}T + (\bar{\nabla} h)(T,T,T).$$

Since γ is a normal section of M at o, by Lemma 3.1, $A_{h(t,t)}t \wedge t = 0$, where $t = T(0)$. In other words,

$$< h(t,t), h(t,t^{\perp}) >= 0,$$

where $t^{\perp}$ is a unit vector tangent to M at o perpendicular to t, which implies that M is isotropic at o. Thus we have

PROPOSITION 2. *Let M be a submanifold of E^m with (*). Then M is isotropic at o.*

Since every geodesic through o is a plane curve, we may represent the immersion $x : M \longrightarrow E^m$ locally on a neighborhood U of o in terms of geodesic polar coordinates $(s, \theta_1, \theta_2 \cdots, \theta_{n-1})$ as

$$\begin{aligned} x(s,\theta_1,\cdots,\theta_{n-1}) =& h(s,\theta_1,\cdots,\theta_{n-1})\mathrm{e}(\theta_1,\cdots,\theta_{n-1}) \\ &+ k(s,\theta_1,\cdots,\theta_{n-1})N(\theta_1,\cdots,\theta_{n-1}), \end{aligned} \tag{3.3}$$

where $\mathrm{e}(\theta_1,\cdots,\theta_{n-1})$ is a unit tangent vector to M at o, h and k functions satisfying $h(0,\theta_1,\cdots,\theta_{n-1}) = k(0,\theta_1,\cdots,\theta_{n-1}) = o$ and $N(\theta_1,\cdots,\theta_{n-1})$ a unit vector normal to M at o depending on $\theta_1,\cdots,\theta_{n-1}$. Then, it is obvious that $\frac{\partial}{\partial\theta_i}\mathrm{e}(\theta_1,\cdots,\theta_{n-1})$ is tangent to M and $\frac{\partial}{\partial\theta_i} N(\theta_1,\cdots,\theta_{n-1})$ normal to M at o for all $i = 1,2,\cdots,n-1$. We then have orthogonal vector fields tangent to M defined on U :

$$\begin{aligned} x_*(\frac{\partial}{\partial s}) =& \frac{\partial h}{\partial s}(s,\theta_1,\cdots,\theta_{n-1})\mathrm{e}(\theta_1,\cdots,\theta_{n-1}) \\ &+ \frac{\partial k}{\partial s}(s,\theta_1,\cdots,\theta_{n-1})N(\theta_1,\cdots,\theta_{n-1}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} x_*(\frac{\partial}{\partial\theta_i}) = & \frac{\partial h}{\partial\theta_i}(s,\theta_1,\cdots,\theta_{n-1})\mathrm{e}(\theta_1,\cdots,\theta_{n-1}) \\ & + h(s,\theta_1,\cdots,\theta_{n-1})\frac{\partial \mathrm{e}}{\partial\theta_i}(\theta_1,\cdots,\theta_{n-1}) \\ & + \frac{\partial k}{\partial\theta_i}(s,\theta_1,\cdots,\theta_{n-1})N(\theta_1,\cdots,\theta_{n-1}) \\ & + k(s,\theta_1,\cdots,\theta_{n-1})\frac{\partial N}{\partial\theta_i}(\theta_1,\cdots,\theta_{n-1}) \end{aligned} \tag{3.5}$$

for $i = 1, 2, \cdots, n-1$, where $x_*(\frac{\partial}{\partial s})(0,\theta_1,\cdots,\theta_{n-1}) = \mathrm{e}(\theta_1,\cdots,\theta_{n-1})$. Since $< x_*(\frac{\partial}{\partial s}), x_*(\frac{\partial}{\partial s}) >= 1$ for each $(s,\theta_1,\cdots,\theta_{n-1})$,

$$\left(\frac{\partial h}{\partial s}\right)^2 + \left(\frac{\partial k}{\partial s}\right)^2 = 1, \tag{3.6}$$

from which, we may put

$$\frac{\partial h}{\partial s}(s,\theta_1,\cdots,\theta_{n-1}) = cos f(s,\theta_1,\cdots,\theta_{n-1}), \tag{3.7}$$

$$\frac{\partial k}{\partial s}(s,\theta_1,\cdots,\theta_{n-1}) = sin f(s,\theta_1,\cdots,\theta_{n-1}), \tag{3.8}$$

where $f(s,\theta_1,\cdots,\theta_{n-1})$ is a smooth function satisfying $f(0,\theta_1,\cdots,\theta_{n-1}) = 0$ for all $\theta_1,\cdots,\theta_{n-1}$.

From (3.4), we have

$$\begin{aligned} \tilde{\nabla}_{x_*(\frac{\partial}{\partial s})}x_*(\frac{\partial}{\partial s}) = & \frac{\partial^2 h}{\partial s^2}\mathrm{e}(\theta_1,\cdots,\theta_{n-1}) \\ & + \frac{\partial^2 k}{\partial s^2}N(\theta_1,\cdots,\theta_{n-1}) \end{aligned} \tag{3.9}$$

which is normal to M since $x(s,\theta_1,\cdots,\theta_{n-1})$ is a geodesic for each $\theta_1,\cdots,\theta_{n-1}$.

We now prove

LEMMA 3. *Let M be a submanifold of Euclidean space satisfying $(*)$. Then the curvature of all the geodesic passing through o is independent of the choice of geodesics.*

Proof. Let γ be a geodesic passing through o. Then $\gamma(s) = x(s,\theta_1,\cdots,\theta_{n-1})$ for some $\theta_1,\cdots,\theta_{n-1}$. The curvature κ of γ is given by

$$(\kappa(s,\theta_1,\cdots,\theta_{n-1}))^2 =< h(T,T), h(T,T) >,$$

where $\gamma'(s) = T$. We compute

$$
\begin{aligned}
&\frac{1}{2}x_*(\frac{\partial}{\partial\theta_i})(\kappa(s_1,\theta_1,\cdots,\theta_{n-1}))^2\\
=&\langle\nabla^{\perp}_{\frac{\partial}{\partial\theta_i}} h(T,T), h(T,T)\rangle\\
=&\langle(\bar{\nabla}h)(\frac{\partial}{\partial\theta_i},T,T), h(T,T)\rangle + 2\langle h(\nabla_{\frac{\partial}{\partial\theta_i}}T,T), h(T,T)\rangle\\
=&\langle(\bar{\nabla}h)(T,\frac{\partial}{\partial\theta_i},T), h(T,T)\rangle + 2\langle h(\nabla_{\frac{\partial}{\partial\theta_i}}T,T), h(T,T)\rangle\\
=&\langle(\bar{\nabla}h)(T,\frac{\partial}{\partial\theta_i},T), h(T,T)\rangle \text{ (because of (3.2))}\\
=&T\langle h(\frac{\partial}{\partial\theta_i},T), h(T,T)\rangle - \langle h(\nabla_T\frac{\partial}{\partial\theta_i},T), h(T,T)\rangle\\
&- \langle h(\frac{\partial}{\partial\theta_i},T), (\bar{\nabla}h)(T,T,T)\rangle\\
=&-\langle(\bar{\nabla}h)(T,T,T), h(\frac{\partial}{\partial\theta_i},T)\rangle \quad \text{(because of (3.2))}
\end{aligned}
$$

for all $i = 1, 2, \cdots, n-1$. Suppose that $< (\bar{\nabla}h)(T,T,T), h(\frac{\partial}{\partial\theta_i},T) >\neq 0$ for some $s_0 \in \mathrm{Dom}\,\gamma$. Then, $(\bar{\nabla}h)(T,T,T) \neq 0$ and $h(\frac{\partial}{\partial\theta_i},T) \neq 0$ at $(s_0,\theta_1,\cdots,\theta_{n-1})$. Thus, $(\bar{\nabla}h)(T,T,T) \neq 0$ and $h(\frac{\partial}{\partial\theta_i},T) \neq 0$ for all $(s,\theta_1,\cdots,\theta_{n-1})$, where s belongs to some open interval J contained in $\mathrm{Dom}\,\gamma$. Since $\gamma'(s)\wedge\gamma''(s)\wedge\gamma'''(s) = 0$ along γ, we see that $h(T,T)\wedge(\bar{\nabla}h)(T,T,T) = 0$. It follows that $h(T,T) = 0$ at $(s,\theta_1,\cdots,\theta_{n-1})$ for every $s \in J$, which is a contradiction. Therefore, $x_*(\frac{\partial}{\partial\theta_i})\,(\kappa(s,\theta_1,\cdots,\theta_{n-1}))^2 = 0$ for all $i = 1,\cdots,n-1$ and all s in $\mathrm{Dom}\,\gamma$. This completes the proof. □

LEMMA 4. *The functions $h(s,\theta_1,\cdots,\theta_{n-1})$ and $k(s,\theta_1,\cdots,\theta_{n-1})$ depend only on s.*

Proof. Taking account of (3.4), (3.7), (3.8) and (3.9), we have

$$(\kappa(s,\theta_1,\cdots,\theta_{n-1}))^2 = (\frac{\partial f}{\partial s})^2,$$

from which,

$$\frac{\partial f}{\partial s} = \varepsilon\kappa(s,\theta_1,\cdots,\theta_{n-1}),\ \varepsilon = \pm 1.$$

According to Lemma 3, $\frac{\partial f}{\partial s}(s,\theta_1,\cdots,\theta_{n-1})$ depends only on s. Since $f(0,\theta_1,\cdots,\theta_{n-1})=0$, we may put

$$f(s,\theta_1,\cdots,\theta_{n-1})=\phi(s).$$

Thus, we have

$$h(s,\theta_1,\cdots,\theta_{n-1})=\int_0^s cos\phi(t)dt,$$
$$k(s,\theta_1,\cdots,\theta_{n-1})=\int_0^s sin\phi(t)dt$$

because of $h(0,\theta_1,\cdots,\theta_{n-1})=k(0,\theta_1,\cdots,\theta_{n-1})=0$. The proof is completed. □

THEOREM 5. *Let M be an n-dimensional complete submanifold of Euclidean space E^m satisfying $(*)$. Then M is a pointed Blaschke manifold or has no cut point for all geodesics through o.*

Proof. From Lemma 3 and 4 we obtain

$$x(s,\theta_1,\cdots,\theta_{n-1})=h(s)\mathrm{e}(\theta_1,\cdots,\theta_{n-1})+k(s)N(\theta_1,\cdots,\theta_{n-1}).$$

If a geodesic passing through o has a cut point, then the cut locus $Cut(o)$ is spherical, that is, the cut value is independent of the choice of the geodesics passing through o. According to [1], pp.138, M is a Blaschke manifold at o. □

COROLLARY 6([9]). *Let M be an n-dimensional complete submanifold of Euclidean space E^m with planar geodesics. Then, M is a (compact) Blaschke manifold or an n-plane E^n.*

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