Bull. Korean Math. Soc. 31 (1994), No. 2, pp. 309–318

EXISTENCE OF SOLUTION OF FINITE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

The mathematical formulation of many problems arising in practice leads to boundary value problems for partial differential equations - especially of parabolic type- with the feature that the boundary is not prescribed in advance but depends on certain properties of the solution itself. Probably the oldest such free boundary problem is due to Stefan (1889). In one space dimension, it may be descrided as follows: In a domain

$$\Omega := \{(y,\tau) \mid \tau > 0, 0 < y < s(\tau)\}$$

a function u is sought as the solution of the heat equatioin

$$u_\tau - u_{yy} = f \qquad \text{in } \Omega$$

The initial temparature $u(y,0)$ with compatiblity conditions as well as the water temparature at $y = 0$:

$$u(0,\tau) = f(\tau) \qquad 0 < \tau \leq T_0$$

are prescribed. The free boundary $y = s(\tau)$ is defined by the condition

$$u(s(\tau),\tau) = 0$$

and the additional condition

$$s_\tau + u_y(s(\tau),\tau) = 0.$$

Received October 20, 1993. Revised November 12, 1993.
This paper is worked on Post-Doc in Institute of Korean Science and Technology.

The melting (or freezing) of an ice block is one of the physical interpretations.

The approach presented in this paper is based on the transformation of the Stefan problem in one space dimension to an initial-boundary value problem for the heat equation in a fixed domain. Of course, the problem is non-linear. The finite element approximation adopted here is the standard continuous Galerkin method in time. In this paper, only the regular case is discussed. This means the error analysis is based on the assumption that the solution is sufficiently smooth.

The aim of this paper is the existence of the solution in a finite Galerkin system of ordinary equations.

2. Weak formulation of the Stefan problem, finite element method

The Stefan problem is Problem P_U: Given $T_0 > 0$, $g(y) \in C^1(0,1)$ with $g(1) = 0$ and $f(\tau) \in C^2(0, T_0]$. Find $\{(s(\tau), U(y,\tau))\}$ such that

$$\begin{aligned}
&s(\tau) > 0 &&\text{for } 0 < \tau \leq T_0\\
&s(0) = 1, &&\\
&U_{yy} - U_\tau = 0 &&\text{in } \Omega = \{(y,\tau) | 0 < \tau \leq T_0, 0 < y < s(\tau)\}\\
&U(0,\tau) = f(\tau) &&\\
&U(s(\tau),\tau) = 0 &&\text{for } 0 < \tau \leq T_0\\
&U(y,0) = g(y) &&\text{for } y \in (0,1)
\end{aligned}$$

and in addition

$$\frac{ds}{d\tau} + U_y(s(\tau),\tau) = 0 \qquad \text{for } 0 < \tau \leq T_0.$$

In order to reduce this problem to one with fixed boundaries, we introduce the new space variable

$$x = s^{-1}(\tau) y. \tag{2.1}$$

The corresponding transformation of τ defined by

$$\frac{d\tau}{dt} = s^2(\tau), \quad \tau(0) = 0 \tag{2.2}$$

leads for the formulation $u(x,t) = U(y,\tau)$ to Problem P_u1: Find $\{s(t), u(x,t), \tau(t)\}$ such that

$$u_{xx} - u_t = xu_x(1,t)u_x(x,t) \qquad \text{in } Q = \{(x,t)|x \in I, 0 < t \leq T\} \tag{2.3}$$

$$\frac{ds}{dt} = -u_x(1,t)s(t), \quad s(0) = 1 \qquad \text{for } 0 < t \leq T \tag{2.4}$$

$$\frac{d\tau}{dt} = s^2(t), \quad \tau(0) = 0 \tag{2.5}$$

with the boundary conditions

$$u(0,t) = f(\tau(t)), \quad u(1,t) = 0 \qquad \text{for } 0 < t \leq T \tag{2.6}$$

and the initial condition

$$u(x,0) = g(x) \qquad \text{for } 0 < x < 1.$$

Here $t = T$ corresponds to $\tau = T_0$. The original Stefan problem is now split into a nonlinear parabolic initial boundary value problem for a fixed domain and two ordinary differential equations (2.4) and (2.5). If the boundary condition at $x = 0$ is time-dependent, the parabolic problem and the ordinary differential equations are coupled.

Let us introduce the space

$$\dot{H}_2 = \{w | \ w \in H_2, w(1) = 0\}$$

Then u belongs to $\dot{H}_2$. Multiplication of (2.3) by w_{xx}, with $w \in \dot{H}_2$ and integration give

$$(u_{xx}, w_{xx}) - (u_t, w_{xx}) = u_x(1,t)(xu_x, w_{xx})$$

here $(\cdot,\cdot)$ denotes the $L_2(0,1)$-scalar product. For simplicity u' and $\dot{u}$ denote differentiation with respect to x and t. Differentiation the equation (2.6) with respect to time t gives

$$\dot{u}(0,t) = \frac{df}{d\tau}(\tau(t))s^2(t)$$
$$\dot{u}(1,t) = 0 \qquad \text{for } 0 < t \le T.$$

In this way we come to Problem P_u2: Find $\{s(t), u(x,t), \tau(t)\}$ such that $u(\cdot,t) \in \dot{H}_2$ and

$$(\dot{u}', w') + (u'', w'') = u'(1,t)(xu', w'') - \frac{df}{d\tau}s^2 w'(0) \tag{2.7}$$

$$\frac{ds}{dt} = -u_x(1,t)s(t), \ \ s(0) = 1 \qquad \text{for } 0 < t \le T \tag{2.8}$$

$$\frac{d\tau}{dt} = s^2(t), \ \ \tau(0) = 0 \tag{2.9}$$

$$\text{for } w \in \dot{H}_2 \text{ and } 0 < t \le T$$

with the initial condition

$$u(x,0) = g(x) \qquad \text{for} \quad 0 < x < 1.$$

(2.7) is the weak formulation of (2.3) and (2.6).

There are many methods for getting finite element solutions. We will study only the standard one: Let $S_h = S_h^{2,r}$, denote the continuous splines of order r with $r \ge 5$, i.e.,S_h consists of continuous piecewise polynomial functions of degree less than r for some regular subdivision Γ_h of (0,1). Up to fixed factors, h is a lower and an upper bound of the length of the subintervals of Γ_h. Furthermore

$$\dot{S}_h = \{w| \ w \in S_h, w(1) = 0\}$$

In this way, $\dot{S}_h \subset \dot{H}_2$. The finite element method is Problem P_{u_h}: Find $\{s_h(t), u_h(x,t), \tau_{h(t)}\}$ such that $u_h(\cdot,t) \in \dot{S}_h$ and

$$(\dot{u}_h', \chi') + (u_h'', \chi'') = u_h'(1,t)(xu_h', \chi'') - \frac{df}{d\tau}s_h^2\chi'(0) \tag{2.10}$$

$$\frac{ds_h}{dt} = -(u_h)_x(1,t)s_h(t), \ \ s_h(0) = 1 \text{ for } 0 < t \le T \tag{2.11}$$

$$\frac{d\tau_h}{dt} = s_h^2(t), \ \ \tau_h(0) = 0 \tag{2.12}$$

$$\text{for } \chi \in \dot{S}_h \text{ and } 0 < t \leq T$$

with the initial condition

$$u_h(\cdot, 0) = P_h g.$$

Here P_h is a projection onto $\dot{S}_h$.

3. Existence of solution

Since S_h is finite dimensional the problem P_{u_h} results in a finite system of ordinary differential equations. Therefore u_h always exists locally i.e., in a certain interval $(0, \bar{t})$, here $\bar{t}$ may depend on g but not on $\dot{S}_h$.

THEOREM. *Problem P_{u_h} has a solution locally.*

Proof. (2.8) with $\chi = u_h$ gives

$$\frac{1}{2}\frac{d}{dt}\|u_h'\|^2 + \|u_h''\|^2 = u_h'(1,t)(xu_h', u_h'') - \frac{df}{d\tau}(\tau_h(t))s_h^2 u_h'(0,t). \quad (3.1)$$

By Mean-value theorem there exists $\psi \in (0,1)$ such that

$$u_h(1,t) - u_h(0,t) = u_h'(\psi, t).$$

On the other hand, from Taylor's theorem

$$u_h(0,t) = u_h(1,t) - \int_0^1 u_h'(x,t)dx.$$

Thus we have

$$u_h'(\psi, t) = \int_0^1 u_h'(x,t)dx.$$

From this we get

$${u_h'}^2(\psi, t) < \|u_h'\|^2. \quad (3.2)$$

By Taylor's theorem we have

$$u_h'^2(1,t) = u_h'^2(\psi,t) + 2\int_\psi^1 u_h' u_h''$$
$$u_h'^2(0,t) = u_h'^2(\psi,t) - 2\int_0^\psi u_h' u_h''.$$

So, we have the inequalities

$$|u_h'(1,t)| \le \|u_h'\| + \sqrt{2}\|u_h'\|^{\frac{1}{2}}\|u_h''\|^{\frac{1}{2}}$$
$$|u_h'(0,t)| \le \|u_h'\| + \sqrt{2}\|u_h'\|^{\frac{1}{2}}\|u_h''\|^{\frac{1}{2}} \tag{3.3}$$

Let $c_1 := \max_{o<t\le T} |\frac{df}{d\tau}(\tau_h(t))|$. Then we have from (3.1) and (3.2)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h'\|^2 + \|u_h''\|^2 \le &(\|u_h'\| + \sqrt{2}\|u_h'\|^{\frac{1}{2}}\|u_h''\|^{\frac{1}{2}})\|u_h'\|\|u_h''\| \\ &+ c_1 s_h^2(\|u_h'\| + \sqrt{2}\|u_h'\|^{\frac{1}{2}}\|u_h''\|^{\frac{1}{2}}).\end{aligned} \tag{3.4}$$

By Young's inequality, we can have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_h'\|^2 + \|u_h''\|^2 \\ \le& 2\|u_h'\|^4 + \frac{1}{8}\|u_h''\|^2 + \|u_h'\|^6 + \frac{3}{4}\|u_h''\|^2 + c_1 s_h^2\|u_h'\| \\ &+ 3c_1^{\frac{4}{3}} s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}} + \frac{1}{64}\|u_h''\|^2 \\ \le& \frac{57}{64}\|u_h''\|^2 + \|u_h'\|^6 + 2\|u_h'\|^4 + c_1 s_h^2\|u_h'\| + 3c_1^{\frac{4}{3}} s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}}.\end{aligned} \tag{3.5}$$

Next, we give the estimate for $(\dot{s_h^2})$. Multiplication (2.8) by $2s_h$ gives

$$(\dot{s_h^2}) = -2u_h'(1,t)s_h^2$$

From (3.3) and Young's inequality, we get

$$\begin{aligned}(\dot{s_h^2}) &\le 2(\|u_h'\| + \sqrt{2}\|u_h'\|^{\frac{1}{2}}\|u_h''\|^{\frac{1}{2}})s_h^2 \\ &\le 2s_h^2\|u_h'\| + \frac{1}{32}\|u_h''\|^2 + 6s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}}.\end{aligned} \tag{3.6}$$

In order to connect (3.5) with (3.6), we introduce a function $\lambda = \lambda(t)$, which is dependent on $\|u_h'\|^2$ and on s_h^2. Define

$$\lambda(t) = \|u_h'\|^2 + s_h^2(t) + 1. \tag{3.7}$$

We can see $\lambda(t) \geq 1$ for all $t \geq 0$. Differentiation (3.7) with respect to t and (3.6) gives

$$\begin{aligned}\dot{\lambda}(t) &= \frac{d}{dt}\|u_h'\|^2 + \frac{d}{dt}s_h^2(t) \\ &\leq \frac{d}{dt}\|u_h'\|^2 + \frac{1}{32}\|u_h''\|^2 + 2s_h^2\|u_h'\| + 6s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}} \\ &\leq \frac{d}{dt}\|u_h'\|^2 + \frac{7}{32}\|u_h''\|^2 + 2s_h^2\|u_h'\| + 6s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}}.\end{aligned} \tag{3.8}$$

Estimating the first two terms of right hand side of (3.8). (3.5) gives

$$\begin{aligned}\dot{\lambda}(t) &\leq 2\|u_h'\|^6 + 4\|u_h'\|^4 + (2c_1 + 2)s_h^2\|u_h'\| + (6c_1^{\frac{4}{3}} + 6)s_h^{\frac{8}{3}}\|u_h'\|^{\frac{2}{3}} \\ &\leq 2\lambda^3 + 4\lambda^2 + (c_1 + 1)^2\lambda^2 + \lambda + (6c_1^{\frac{4}{3}} + 6)\lambda^{\frac{5}{3}}.\end{aligned} \tag{3.9}$$

The second inequality follows from construction of λ in (3.7). Since $\lambda(t) \geq 1$, (3.9) gives

$$\dot{\lambda}(t) \leq \bar{c}\lambda^3 \qquad \text{with } \bar{c} = 7 + (c_1 + 1)^2 + (6c_1^{\frac{4}{3}} + 6). \tag{3.10}$$

For the function λ, the initial condition

$$\lambda(0) = \lambda_0 = \|P_h g'\|^2 + 2 \tag{3.11}$$

holds. For this function λ with (3.10) and (3.11) we construct a function $\mu = \mu(t)$, which satisfies the following integral inequality

$$\dot{\mu}(t) \geq \bar{c}\mu^3, \qquad \mu(0) \geq \lambda_0. \tag{3.12}$$

By monotone increasing kernel theorem and from (3.10) and (3.12),we have $0 \leq \lambda \leq \mu$. We give an explicit function $\mu = \mu(t)$ and prove the inequalities (3.12). Put

$$\mu(t) = \frac{1}{4}\lambda_0\Big(\frac{1}{25} - \bar{c}\frac{1}{7}\lambda_0^2 t\Big)^{-\frac{1}{2}}. \tag{3.13}$$

Then

$$\mu(0) = \frac{5}{4}\lambda_0 \geq \lambda_0$$

and

$$\begin{aligned} \dot{\mu}(t) =& \frac{1}{2}\bar{c}\frac{1}{7}\lambda_0^2\frac{1}{4}\lambda_0\Big(\frac{1}{25} - \bar{c}\frac{1}{7}\lambda_0^2 t\Big)^{-\frac{3}{2}} \\ =& \frac{1}{56}\lambda_0^3\bar{c}\Big(\frac{1}{25} - \bar{c}\frac{1}{7}\lambda_0^2 t\Big)^{-\frac{3}{2}} \\ \geq& \frac{1}{64}\lambda_0^3\bar{c}\Big(\frac{1}{25} - \bar{c}\frac{1}{7}\lambda_0^2 t\Big)^{-\frac{3}{2}} = \bar{c}\mu^3(t) \quad \text{for } t < \frac{7}{25\lambda_0^2\bar{c}}. \end{aligned} \tag{3.14}$$

But the condition on t does not mean the restriction, since in relation to the existence of solution of problem P_{u_h} we only deal with a small interval. With (3.13) we have also a solution of integral inequalities (3.12). Since the continuous function μ is bounded on the closed interval $0 \leq t \leq \frac{1}{5\lambda_0^2\bar{c}}$, every solution of integral inequality (3.10) is bounded on the closed interval $0 \leq t \leq \frac{1}{5\lambda_0^2\bar{c}}$. On the ground of the construction (3.7) we have the bounds of $\|u_h'\|^2$ and s_h^2. Let

$$c_s = \max_{0<t\leq\frac{1}{5\lambda_0^2\bar{c}}} s_h^2(t).$$

Then (3.5) gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_h'\|^2 + \|u_h''\|^2 \leq& 2\|u_h'\|^4 + \frac{1}{8}\|u_h''\|^2 + \|u_h'\|^6 + \frac{3}{4}\|u_h''\|^2 \\ &+ c_1c_s\|u_h'\| + 3(c_1c_s)^{\frac{4}{3}}\|u_h'\|^{\frac{2}{3}} + \frac{1}{64}\|u_h''\|^2 \end{aligned} \tag{3.15}$$

Since the last term is smaller than $\frac{7}{64}\|u_h''\|^2$, $\|u_h''\|^2$ terms in both sides of (3.15) vanishe and we get

$$\begin{aligned} \frac{d}{dt}\|u_h'\|^2 \leq& \frac{1}{3} + \frac{16}{3}\|u_h'\|^6 + 2\|u_h'\|^6 + \frac{5}{6}(\sqrt{2}c_1c_s)^{\frac{6}{5}} \\ &+ \frac{4}{3}\|u_h'\|^6 + \frac{8}{9}(6(c_1c_s)^{\frac{4}{3}})^{\frac{9}{8}} + \frac{1}{9}\|u_h'\|^6 \\ =& \frac{79}{9}\|u_h'\|^6 + \hat{c} \end{aligned} \tag{3.16}$$

with

$$\hat{c} = \frac{1}{3} + \frac{5}{6}(2^{\frac{1}{2}} c_1 c_s)^{\frac{6}{5}} + \frac{8}{9}(6(c_1 c_s)^{\frac{4}{3}})^{\frac{9}{8}}.$$

There are two cases. First we consider the case of $\|u_h'\| < 1$. From existence theorem of heat equation there exists a solution. Second in case of $\|u_h'\| \geq 1$, from (3.16) we have

$$\begin{aligned} \frac{d}{dt}\|u_h'\|^2 &\leq \frac{79}{9}\|u_h'\|^6 + \hat{c} \qquad (3.17) \\ &\leq \max\{\frac{79}{9} + \hat{c}, \frac{5}{2}\bar{c}(\|P_h g'\|^3 + 4\|P_h g'\| + \frac{4}{\|P_h g'\|})\|u_h'\|^6\} \\ &=: c'\|u_h'\|^6. \end{aligned}$$

For simplicity let $h(t) := \|u_h'\|^2$, then

$$\frac{d}{dt}h(t) \leq c' h^3(t). \qquad (3.18)$$

Integration (3.18) gives

$$h^2(t) \leq \frac{h^2(0)}{1 - 2c'th^2(0)} \qquad \text{for } t < \frac{1}{2c'h^2(0)}.$$

So, we have

$$\|u_h'\|^2 \leq \frac{\|P_h g'\|^2}{(1 - 2c't\|P_h g'\|^4)^{\frac{1}{2}}} \qquad \text{for } t < \frac{1}{2c'\|P_h g'\|^4}.$$

Since g is sufficiently smooth, we can assume $\|P_h g'\|$ is uniformly bounded i.e., $\|P_h g'\| \leq c(g)$. With this assumption the Existence theorem of heat equation guarantees the existence not only of u_h' and but also of u_h for all t with

$$t < \frac{1}{2c'c(g)}. \qquad (3.19)$$

From definition of c' in (3.17) and because of (3.19),

$$t \leq \frac{1}{5\lambda_0^2\bar{c}} < \frac{7}{25\lambda_0^2\bar{c}}.$$

Therefore the condition on t in (3.14) is satisfied. Hence the problem P_{u_h} has a local solution.

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