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SPACE CURVES SATISFYING $\Delta H = AH$

DONG-SOO KIM AND HEI-SUN CHUNG

1. Introduction

Let $x : M^n \to \mathbf{E}^m$ be an isometric immersion of a manifold M^n into the Euclidean space $\mathbf{E}^m$ and Δ the Laplacian of M^n defined by $-div \circ grad$. The family of such immersions satisfying the condition $\Delta x = \lambda x, \lambda \in \mathbf{R}$, is characterized by a well known result of Takahashi ([8]): they are either minimal in $\mathbf{E}^m$ or minimal in some Euclidean hypersphere.

As a generalization of Takahashi's result, many authors ([3,6,7]) studied the hypersurfaces $\mathbf{M}^n$ in $\mathbf{E}^{n+1}$ satisfying $\Delta x = Ax + b$, where A is a square matrix and b is a vector in $\mathbf{E}^{n+1}$, and they proved independently that such hypersurfaces are either minimal in $\mathbf{E}^{n+1}$ or hyperspheres or spherical cylinders.

Since $\Delta x = -nH$, the submanifolds mentioned above satisfy $\Delta H = \lambda H$ or $\Delta H = AH$, where H is the mean curvature vector field of M. And the family of hypersurfaces satisfying $\Delta H = \lambda H$ was explored for some cases in [4].

In this paper, we classify space curves $x : R \to \mathbf{E}^3$ satisfying $\Delta x = Ax + b$ or $\Delta H = AH$, and find conditions for such curves to be equivalent.

2. Space curves satisfying $\Delta x = Ax + b$

Let $x : R \to \mathbf{E}^m$ be a curve parametrized by arclength s. Then the Laplacian Δ of x is given by $\Delta = -\partial^2/\partial s^2$ and the mean curvature

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vector field is given by $H = x''(s)$. If x satisfies $\Delta x = Ax + b$ or $\Delta H = AH$, then x can be written as

$$x(s) = a_0 + as + \sum_{i=1}^{k}\{b_i \cos(l_i s) + c_i \sin(l_i s)\} \tag{$*$}$$

where $l_1 < l_2 < \cdots < l_k$ are positive real numbers and a_0, a, b_i, c_i are vectors in $\mathbf{E}^m$ such that for each $i \in \{1, \cdots, k\}$, b_i and c_i are not simultaneously zero ([3]).

Note that the condition $\langle x'(s), x'(s)\rangle \equiv 1$ is equivalent to the following ([1]) :

$$\sum_{i=1}^{k} l_i^2\, D_{ii} = 2(1 - |a|^2), \tag{2.1}$$

$$-4\sum_{\substack{i=1\\ l_i=l}}^{k} l_i M_i + \sum_{\substack{i=1\\ 2l_i=l}}^{k} l_i^2 A_{ii} + 2\sum_{\substack{i>j\\ l_i+l_j=l}} l_i l_j A_{ij} - 2\sum_{\substack{i>j\\ l_i-l_j=l}} l_i l_j D_{ij} = 0, \tag{2.2}$$

$$4\sum_{\substack{i=1\\ l_i=l}}^{k} l_i \overline{M}_i + \sum_{\substack{i=1\\ 2l_i=l}}^{k} l_i^2 \overline{A}_{ii} + 2\sum_{\substack{i>j\\ l_i+l_j=l}} l_i l_j \overline{A}_{ij} + 2\sum_{\substack{i>j\\ l_i-l_j=l}} l_i l_j \overline{D}_{ij} = 0, \tag{2.3}$$

for all $l \in \{l_i | 1 \le i \le k\} \cup \{l_i + l_j | 1 \le j \le i \le k\} \cup \{l_i - l_j | 1 \le j < i \le k\}$, where

$$\begin{aligned} M_i &= \langle a, c_i\rangle, & \overline{M}_i &= \langle a, b_i\rangle, \\ A_{ij} &= \langle b_i, b_j\rangle - \langle c_i, c_j\rangle, & D_{ij} &= \langle b_i, b_j\rangle + \langle c_i, c_j\rangle, \\ \overline{A}_{ij} &= \langle b_i, c_j\rangle + \langle b_j, c_i\rangle, & \overline{D}_{ij} &= \langle b_i, c_j\rangle - \langle b_j, c_i\rangle, \end{aligned} \tag{2.4}$$

for all $i, j \in \{1, 2, \cdots, k\}$.

For $l = 2l_k$ we obtain from (2.2) and (2.3) that

$$A_{kk} = \overline{A}_{kk} = 0. \tag{2.5}$$

Thus b_k and c_k are orthogonal and have the same length. If we define E_0 and E_i, $1 \le i \le k$, as the subspace of E^m generated by a and $\{b_i, c_i\}$, respectively, then we see that $n_0 \in \{0,1\}$, $n_i \in \{1,2\}$, $i = 1, \cdots, k-1$ and $n_k = 2$, where $n_i = \dim E_i$ for $i \in \{0, 1, \cdots, k\}$.

Now suppose that $\Delta x(s) = Ax(s) + b$ for some square matrix A and a vector b in $\mathbf{E}^m$. Then we have

$$Aa = 0, \quad Ab_i = l_i^2 b_i, \quad Ac_i = l_i^2 c_i. \tag{2.6}$$

Hence the subspaces $E_0, E_1, \cdots, E_k$ are linearly independent. Thus we have $n_0 + n_1 + \cdots + n_k \leq m$.

THEOREM A. *A space curve $x : R \to \mathbf{E}^3$ satisfies $\Delta x = Ax + b$ if and only if $x(s)$ is either a straight line or a circle or a circular helix.*

Proof. Since $n_0 + n_1 + \cdots + n_k \leq 3$, we have $k \leq 2$.

Case 1) $k = 0$. In this case, $n_0 = 1$ and $|a| = 1$. Hence $x(s) = a_0 + as$ is a straight line.

Case 2) $k = 1$. In this case, (2.5) shows that b_1 and c_1 are orthogonal and have the same length r. Hence $D_{11} = 2r^2$ and from (2.1) we have $|a|^2 + l_1^2 r^2 = 1$. And for $l = l_1$ (2.2) and (2.3) show that $M_1 = \overline{M}_1 = 0$. Thus a is orthogonal to E_1. Hence it can be easily shown that for $a = 0$ $x(s)$ is a circle and that for $a \neq 0$ $x(s)$ is a circular helix.

Case 3) $k = 2$. In this case, $n_0 = 0, n = 1$ and $n_2 = 2$. Hence we have

$$x(s) = a_0 + \sum_{i=1}^{2} \{b_i \cos(l_i s) + c_i \sin(l_i s)\}.$$

And (2.5) shows that b_2 and c_2 are orthogonal and have the same length r_2. Hence we see that, with a suitable choice of Euclidean coordinates of $\mathbf{E}^3$, we may assume that $b_2 = (r_2, 0, 0), c_2 = (0, r_2, 0)$. For $l = l_1 + l_2$ we obtain from (2.2) and (2.3) that $A_{12} = \overline{A}_{12} = 0$. From these, since b_1 and c_1 are linearly dependent, we see that $b_1 = (0, 0, r_1)$ and $c_1 = (0, 0, \overline{r}_1)$ for some $r_1, \overline{r}_1 \in R$. Now there are three possibilities: 1) $l_2 = 2l_1$, 2) $l_2 = 3l_1$, 3) $l_2 \neq 2l_1$ and $l_2 \neq 3l_1$. In any case (2.2) and (2.3) show that $A_{11} = \overline{A}_{11} = 0$. Hence $b_1 = c_1 = 0$. This contradiction completes the proof of the only if part.

The converse is obvious.

REMARK. Straight lines, circles and circular helices are all the W-curves in $\mathbf{E}^3$ ([1,5]). Hence Theorem A can be restated as follows: A space curve $x : R \to \mathbf{E}^3$ satisfies $\Delta x = Ax + b$ if and only if $x(s)$ is a W-curve.

3. Space curves satisfying $\Delta H = AH$

In this section we consider the curves $x : R \to \mathbf{E}^3$ satisfying $\Delta H = AH$. Such curves can be written as $(*)$. Since $H = x''(s)$, from $\Delta H = AH$ we see that $Ab_i = l_i^2 b_i$ and $Ac_i = l_i^2 c_i$. Hence $E_1, \cdots, E_k$ are linearly independent so that $n_1 + \cdots + n_k \leq 3$. Note that $n_k = 2$, hence $k \leq 2$. If $k = 0$, then $n_0 = 1$ and $x(s)$ is a straight line. Suppose that $k = 1$. Then it can be shown as in the proof of Theorem A that for $a = 0$ $x(s)$ is a circle and for $a \neq 0$ $x(s)$ is a circular helix.

Now suppose that $k = 2$. Then we have $n_1 = 1$ and $n_2 = 2$. Note that b_1 and c_1 are linearly dependent and that $|b_2| = |c_2| = r_2 > 0$ and $\langle b_2, c_2 \rangle = 0$. Hence we may assume that $b_2 = (r_2, 0, 0)$, $c_2 = (0, r_2, 0)$.

Case 1) $l_2 \neq 2l_1$ and $l_2 \neq 3l_1$.

For $l = 2l_1$ (2.2) and (2.3) imply that $|b_1| = |c_1| > 0$ and $\langle b_1, c_1 \rangle = 0$. Since $n_1 = 1$, this is a contradiction.

Case 2) $l_2 = 3l_1$.

For $l = l_1 + l_2$ (2.2) and (2.3) imply that $A_{21} = \overline{A}_{21} = 0$. Since b_1 and c_1 are linearly dependent, these show that $b_1 = (0, 0, r_1)$ and $c_1 = (0, 0, \overline{r}_1)$ for some $r_1, \overline{r}_1 \in R$. For $l = 2l_1 = l_2 - l_1$ (2.2) and (2.3) show that $A_{11} = 6D_{21}$ and $\overline{A}_{11} = -6\overline{D}_{21}$. Since $D_{21} = \overline{D}_{21} = 0$, we have $b_1 = c_1 = 0$. This is a contradiction.

Case 3) $l_2 = 2l_1 (= \frac{2}{r})$.

For $l = l_1 + l_2$, as in Case 2), (2.2) and (2.3) imply that $b_1 = (0, 0, r_1)$ and $c_1 = (0, 0, \overline{r}_1)$ for some $r_1, \overline{r}_1 \in R$ such that r_1 and $\overline{r}_1$ are not simultaneously zero. Since $D_{21} = \overline{D}_{21} = 0$, for $l = l_1$ (2.2) and (2.3) show that $M_1 = \overline{M}_1 = 0$. Hence we have $a = (a_1, a_2, 0)$ for some $a_1, a_2 \in R$. For $l = 2l_1 = l_2$ (2.2) and (2.3) imply that $M_2 = \frac{1}{8r} A_{11}$ and $\overline{M}_2 = -\frac{1}{8r} \overline{A}_{11}$. Hence we have

$$a_1 = -\frac{r_1 \overline{r}_1}{4rr_2}, \quad a_2 = \frac{r_1^2 - \overline{r}_1^2}{8rr_2}. \tag{3.1}$$

From (2.1) we see that

$$r = \frac{r_1^2 + \overline{r}_1^2 + 16r_2^2}{8r_2}. \tag{3.2}$$

Thus for some constants $r_1, \overline{r}_1$ and $r_2 > 0$ such that r_1 and $\overline{r}_1$ are not simultaneously zero, $x(s)$ is of the following form

$$\begin{aligned} x(s) =& a_0 + as + b_1 \cos\frac{s}{r} + c_1 \sin\frac{s}{r} + b_2 \cos\left(\frac{2s}{r}\right) + c_2 \sin\left(\frac{2s}{r}\right) \\ =& a_0 + \left(a_1 s + r_2 \cos\left(\frac{2s}{r}\right), a_2 s + r_2 \sin\left(\frac{2s}{r}\right), r_1 \cos\frac{s}{r} + \overline{r}_1 \sin\frac{s}{r}\right), \end{aligned} \tag{3.3}$$

where a_1, a_2 and r are determined by (3.1) and (3.2).

Hence the following theorem is proved.

THEOREM B. *For any constants $r_1, \overline{r}_1$ and $r_2 > 0$ such that r_1 and $\overline{r}_1$ are not simultaneously zero, $x(s)$ defined by (3.3) satisfies $\Delta H = AH$.*

Conversely, if a curve $x(s)$ in $\mathbf{E}^3$ which is not a W-curve satisfies $\Delta H = AH$, then , up to a Euclidean motion of $\mathbf{E}^3$, $x(s)$ is of the form (3.3).

COROLLARY. *A closed curve $x : R \to E^3$ satisfies $\Delta H = AH$ if and only if x is a circle.*

4. Equivalences of space curves

Let $x : R \to \mathbf{E}^m$ and $y : R \to \mathbf{E}^m$ be two nonclosed curves parametrized by arclength. Then we call x and y are equivalent if and only if there exists a similarity transformation $F : \mathbf{E}^m \to \mathbf{E}^m : x \to \rho Ax + b$ and a mapping $f : R \to R$ determined by $f(s) = \frac{\varepsilon}{\rho}s - d$ ($\rho > 0, A \in O(m), b \in R^m, \varepsilon \in \{-1, 1\}, d \in R$) such that $y = F \circ x \circ f$ ([1]). Roughly speaking, x and y are equivalent if and only if there is a similarity transformation of $\mathbf{E}^m$ mapping the image of x to the image of y.

As in the proof of Lemma 4.2 in [1], we may prove the following lemma.

LEMMA. *Let*

$$x(s) = a_0 + as + \sum_{i=1}^{k} \{a_i \cos(l_i s) + b_i \sin(l_i s)\},$$
$$y(s) = a'_0 + a's + \sum_{i=1}^{k'} \{a'_i \cos(l'_i s) + b'_i \sin(l'_i s)\}$$

be two nonclosed curves of the form $(*)$ *parametrized by arclength. If* x *and* y *are equivalent, then*

(1) $|a| = |a'|$,
(2) $k = k', \frac{l_1}{l'_1} = \frac{l_2}{l'_2} = \cdots = \frac{l_k}{l'_k} (= \rho)$
(3) *there is a positive constant* c *such that for all* $i, j \in \{1, 2, \cdots, k\}$ *with* $i \geq j$

$$A'^2_{ij} + \overline{A}'^2_{ij} = c[A^2_{ij} + \overline{A}_{ij}^2],$$

$$D'^2_{ij} + \overline{D}'^2_{ij} = c[D^2_{ij} + \overline{D}_{ij}^2],$$

where the constants c *and* ρ *are related by* $c = \rho^4$.

Now let $x(s) = x_{r_1 \overline{r}_1 r_2}(s)$ be a curve of the form (3.3), then we have

$$A_{11} = r_1^2 - \overline{r}_1^2,\ D_{11} = r_1^2 + \overline{r}_1^2, \overline{A}_{11} = 2r_1\overline{r}_1,\ D_{22} = 2r_2^2 \tag{4.1}$$

and otherwise zero. And suppose that another curve $y(s) = y_{r'_1 \overline{r}'_1 r'_2}(s)$ of the form (3.3) is equivalent to $x(s)$. Then from the lemma and (4.1) we see that

$$\frac{\sqrt{r'^2_1 + \overline{r}'^2_1}}{\sqrt{r_1^2 + \overline{r}_1^2}} = \frac{r'_2}{r_2} (= \rho). \tag{4.2}$$

Conversely, suppose that two curves $x = x_{r_1 \overline{r}_1 r_2}$ and $y = y_{r'_1 \overline{r}'_1 r'_2}$ of the form (3.3) satisfy the condition (4.2). Then from (3.2) we aslo have $r' = \rho r$. If we let $(r_1, \overline{r}_1) = \delta(\cos\theta, \sin\theta)$, $\delta > 0$, then we have $(r'_1, \overline{r}'_1) = \delta\rho(\cos\theta', \sin\theta')$.

For the constant d satisfying $\frac{d}{r} = (\theta' - \theta)$ let A be the matrix defind by

$$A = \begin{pmatrix} \cos\frac{2d}{r} & -\sin\frac{2d}{r} & 0 \\ \sin\frac{2d}{r} & \cos\frac{2d}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it can be shown that $y(s) = \rho A x(\frac{s}{\rho} - d) + b$, where $b = a_0' - \rho A a_0 + d\rho A a$.

Hence the following theorem is proved.

THEOREM C. *Let $x = x_{r_1\bar{r}_1 r_2}$ and $y = y_{r_1'\bar{r}_1' r_2'}$ be two curves of the form (3.3). Then x and y are equivalent if and only if there is a positive constant ρ satisfying*

$$r_1'^2 + \bar{r}_1'^2 = \rho^2(r_1^2 + \bar{r}_1^2) \text{ and } r_2' = \rho r_2.$$

REMARK. In the terminology of finite type submanifolds, a space curve in E^3 satisfying $\Delta x = Ax + b$ or $\Delta H = AH$ is null 3-type. Closed 2-type space curves were classified by B.Y. Chen, F. Dillen and L. Verstraelen in [2].

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DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, CHONNAM NATIONAL UNIVERSITY, KWANGJU 500-757, KOREA