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TOPOLOGICALLY FREE ACTIONS AND PURELY INFINITE C^*–CROSSED PRODUCTS

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1. Introduction

For a given C^*-dynamical system (A, G, α) with a G-simple C^*-algebra A (that is A has no proper α-invariant ideal) many authors have studied the simplicity of a C^*-crossed product $A \rtimes_\alpha G$. In [1] topological freeness of an action is shown to guarantee the simplicity of the reduced C^*-crossed product $A \rtimes_{\alpha r} G$ when A is G-simple.

In this paper we investigate the pure infiniteness of a simple C^*-crossed product $A \rtimes_\alpha G$ of a purely infinite simple C^*-algebra A and a topologically free action α of a finite group G, and find a sufficient condition in terms of the action on the spectrum of the multiplier algebra $M(A)$ of A. Showing this we also prove that some extension of a topologically free action is still topologically free.

2. Topologically free action

Let (A, G, α) be a C^*-dynamical system. Then there is an action of G on the spectrum $\hat{A}$ of A; for each $\pi \in \hat{A}, t\pi(a) = \pi(\alpha_t(a)), t \in G, a \in A$. An action α is said to be *topologically free* if for any $t_1, \dots, t_n \in G \setminus \{e\}$ the set $\bigcap_{i=1}^n \{\pi \in \hat{A} \mid t_i\pi \neq \pi\}$ is dense in $\hat{A}$.

REMARK 1. If A is simple, then $\hat{A}$ is the only nonempty open set of $\hat{A}$. This is because each open set in $\hat{A}$ corresponds to an ideal of A. Hence, for A, α is topologically free if and only if $\bigcap_{i=1}^n \{\pi \in \hat{A} \mid t_i\pi \neq \pi\}$ is nonempty for every $t_1, \dots, t_n \in G \setminus \{e\}$.

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In [1], it was shown that if α is topologically free, then each automorphism α_t is properly outer for $t \setminus \{e\}$; that is, for every nonzero α_t-invariant ideal I of A and inner automorphism β of I, $\| \alpha_t|_I - \beta\| = 2$. Hence, if α is a topologically free action on a simple C^*-algebra A by a group G, then each automorphism α_t is outer for $t \setminus \{e\}$. Conversely, as it was mentioned in [1], if A is a separable simple C^*-algebra, and each $\alpha_t(t \neq e)$ is outer, then α is topologically free. This follows because topological freeness is weaker than the strong Connes spectrum condition used in [8], and the spectrum condition is equivalent to the outerness of α for simple C^*-algebras.

Let (A, G, α) be a C^*-dynamical system with a discrete group G. Then the action α uniquely extends to an action on the multiplier algebra $M(A)$ of A, and we write this extension by α again.

Recall that a C^*-algebra A is said to be *purely infinite* if every hereditary C^*-subalgebra of A has an infinite projection, a projection equivalent to its subprojection. Obviously, every hereditary C^*-subalgebra of a purely infinite C^*-algebra is purely infinite. It is not known whether a simple C^*-algebra containing an infinite projection is purely infinite or not. For properties and examples of purely infinite C^*-algebras, refer to [3], [12], and [13].

If A is purely infinite, then so is $M(A)$, and the following is proved by Rørdam:

PROPOSITION 2. *[12] Let A be a unital simple C^*-algebra, and let $\mathcal{K}$ denote the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. Then $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple if and only if A is the matrix algebra $M_n(\mathbb{C})$ or purely infinite.*

PROPOSITION 3. *Let (A, G, α) be a C^*-dynamical system where A is a σ-unital nonunital purely infinite simple C^*-algebra. If α is topologically free, then so is the extension α on $M(A)$.*

Proof. For $t_1, \ldots, t_n \in G \setminus \{e\}$, the set $X = \bigcap_{i=1}^n \{\pi \in \hat{A} \,|\, t_i\pi \neq \pi\}$ is dense in $\hat{A}$. Each representation $\pi \in X$ on a Hilbert space H extends uniquely to $M(A)$ [4, Proposition 2.10.4] on H so that the extension, also denoted by π, is still irreducible.

Let T be a bounded operator on H intertwining π and $t_i\pi$; that is,

$T\pi(x) = t_i\pi(x)T$ for $x \in M(A)$. Then T automatically intertwines π and $t_i\pi$ on A. Hence, $T \equiv 0$ because π and $t_i\pi$ are disjoint [9, Corollary 3.13.3]. Therefore, π and $t_i\pi$ are disjoint as representations of $M(A)$ so that $Y = \bigcap_{i=1}^{n}\{\pi \in M(A)\hat{}\ | t_i\pi \neq \pi\}$ has nonempty intersection with $\hat{A}$. Since A is nonunital, A is stable [14, Theorem 1.2] and we hence have $A \cong A\otimes\mathcal{K} \cong A_0\otimes\mathcal{K}$ for each unital hereditary (simple) C^*-subalgebra A_0 of A since a σ-unital simple C^*-algebra is stably isomorphic to its hereditary C^*-subalgebra [2, Corollary 2.6]. Therefore, A is the unique ideal in $M(A)$ by Proposition 2; that is, $\hat{A}$ is the unique proper open subset in $M(A)\hat{}$, and we conclude that Y is dense in $M(A)^\wedge$, and we conclude that Y is dense in $M(A)^\wedge$.

Given a C^*-dynamical system (A, G, α), we have an induced C^*-dynamical system $(A\otimes\mathcal{K}, \tilde{\alpha} = \alpha\otimes\mathrm{id},\ G)$. For a unital simple C^*-algebra A, we will show that $\tilde{\alpha}$ is topologically free if α is. If $\pi : A \to B(H)$ is an irreducible representation of A, then $\tilde{\pi} = \pi\otimes\mathrm{id}\ : A\otimes\mathcal{K} \to B(H\otimes H')$ is still irreducible [7, Proposition 11.3.2], where H' is a separable infinite dimensional Hilbert space. Note that given a C^*-dynamical system (A, G, α), we have the action of G on the spectrum $(A\otimes\mathcal{K})^\wedge$ given by $\tilde{t}\tilde{\pi}(x) = \tilde{\pi}(\tilde{\alpha}_t(x))$ for $x \in A\otimes\mathcal{K}, t \in G$, and $\tilde{\pi} \in (A\otimes\mathcal{K})^\wedge$.

LEMMA 4. *If $\pi : A \to B(H)$ is an irreducible representation of a unital C^*-algebra A such that $t\pi \neq \pi$, then $\tilde{t}\tilde{\pi} \neq \tilde{\pi}, t \in G\setminus\{e\}$.*

Proof. We show that $\tilde{t}\tilde{\pi}$ is disjoint with $\tilde{\pi}$. Let T be an intertwining operator of $\tilde{t}\tilde{\pi}$ and $\tilde{\pi}$ on $H\otimes H'$, that is, $\tilde{\pi}(x)T = T\tilde{t}\tilde{\pi}(x)$ for $x \in A\otimes\mathcal{K}$.

For a nonzero vector $\xi_0 \in H'$, let $p_0 \in \mathcal{K}$ be the projection onto the one dimensional subspace $\langle\xi_0\rangle$ of H' so that $1\otimes p_0 \in A\otimes\mathcal{K}$.

For each vector $\eta\otimes\xi_0 \in H\otimes\xi_0$, we have

$$\begin{aligned}(1\otimes p_0)T(\eta\otimes\xi_0) &= \tilde{\pi}(1\otimes p_0)T(\eta\otimes\xi_0)\\ &= T\tilde{t}\tilde{\pi}(1\otimes p_0)(\eta\otimes\xi_0)\\ &= T\tilde{\pi}(\alpha_t(1)\otimes p_0)(\eta\otimes\xi_0)\\ &= T(\eta\otimes\xi_0).\end{aligned}$$

Let $\{\xi_i\}_{i=0}^{\infty}$ be an orthogonal basis of H'. Then we can write

$$T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0 + \sum_{i=1}^{\infty} \eta_i \otimes \xi_i$$

for some $\eta_i \in H$ and $(1 \otimes p_0)T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0$. Hence, $T(\eta \otimes \xi_0) = \eta_0 \otimes \xi_0$ for some $\eta_0 \in H$. Moreover, it is not difficult to show that the map $T_0 : H \to H$ defined by $T_0(\eta) = \eta_0$ is bounded linear. For $a \otimes p_0 \in A \otimes \mathcal{K}, a \in A$, we have

$$\begin{aligned}
\pi(a)\eta_0 \otimes \xi_0 =&(\pi(a) \otimes p_0)(\eta_0 \otimes \xi_0) = (\pi(a) \otimes p_0)(T(\eta \otimes \xi_0)) \\
=&\tilde{\pi}(a \otimes p_0)T(\eta \otimes \xi_0) = T\tilde{t}\tilde{\pi}(a \otimes p_0)(\eta \otimes \xi_0) \\
=&T(\pi(\alpha_t(a)) \otimes p_0)(\eta \otimes \xi_0) \\
=&T(\pi(\alpha_t(a)\eta \otimes \xi_0) = (\pi(\alpha_t(a))\eta)_0 \otimes \xi_0
\end{aligned}$$

Hence, $\pi(a)\eta_0 = (\pi(\alpha_t(a))\eta)_0$; that is, $\pi(a)T_0 = T_0(t\pi(a))$ for every $a \in A$ because the map $H \to H \otimes \xi_0$ given by $\eta \mapsto \eta \otimes \xi_0$ is injective. Since π and $t\pi$ are disjoint, we conclude that $T_0 \equiv 0$. Therefore, $T \equiv 0$ because ξ_0 is an arbitrary non zero vector of H'.

THEOREM 5. *Let $(A \otimes \mathcal{K}, G, \tilde{\alpha} = \alpha \otimes id)$ be a C^*-dynamical system where A is a simple C^*-algebra. Then $\tilde{\alpha}$ is topologically free if so is α.*

Proof. Since $A \otimes \mathcal{K}$ is simple, it suffices to show that for every $t_1, \dots, t_n \in G \setminus \{e\}$, the set $\bigcap_{i=1}^{n} \{ \tilde{\pi} \in (A \otimes \mathcal{K})^\wedge \mid \tilde{t}_i\tilde{\pi} \neq \tilde{\pi} \}$ is nonempty. But this is almost obvious by Lemma 4.

3. Purely infinite simple C^*-crossed products

Let (A, G, α) be a C^*-dynamical system with a purely infinite simple C^*-algebra A and a finite group G.

In this section, we examine the pure infiniteness of the infinite C^*-algebra $A \rtimes_\alpha G$.

THEOREM 6. *Let (A, G, α) be a C^*-dynamical system with a purely infinite simple C^*-algebra A and a finite group G. If α is a topologically free action such that*

$$\bigcap_{t\in G\setminus\{e\}} \{\, \tilde{\pi} \in (M(A\otimes\mathcal{K}))^\wedge \,|\, \tilde{t}\tilde{\pi} \neq \tilde{\pi} \,\} \cap \hat{A}^{\mathsf{c}} \neq \emptyset,$$

then the infinite simple C^*-crossed product $A \rtimes_\alpha G$ is purely infinite.

Proof. It is known that the fixed point algebra A^α can be regared as a hereditary C^*-subalgebra of the infinite simple C^*-algebra $A \rtimes_\alpha G$ whenever G is compact [10]; hence, A^α contains a projection p [5]. The unital hereditary C^*-subalgebra A_p of A generated by p is invariant under α. Moreover, from [11, Lemma 3.4], we see that if $A_p \rtimes_\alpha G$ is purely infinite, then so is $A \rtimes_\alpha G$. Actually, for each hereditary C^*-subalgebra B of $A \rtimes_\alpha G$, we can find a unitary element $u \in M(A \rtimes_\alpha G)$ such that $uBu^* \cap (A_p \rtimes_\alpha G) \neq 0$; henece, B has an infinite projection if $A_p \rtimes_\alpha G$ is purely infinite. Hence, we may assume that A is unital. Note that $A \rtimes_\alpha G$ is purely infinite if and only if $(A \rtimes_\alpha G)\otimes\mathcal{K}$ is purely infinite [14, Proposition 1.4]. Since $(A \rtimes_\alpha G)\otimes\mathcal{K} \cong (A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G$ [6, Theorem. 2.6] it suffices to show that $M((A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G)/((A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G)$ is simple by Proposition 2. Note that $M((A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G) = M(A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G)$ (G is finite) and

$$M((A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G)/((A\otimes\mathcal{K}) \rtimes_{\tilde{\alpha}} G) \cong (M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})) \rtimes_{\tilde{\alpha}} G.$$

The fact that $M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ is simple implies that the C^*-algebra $(M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})) \rtimes_{\tilde{\alpha}} G$ is simple if $\tilde{\alpha}$ is topologically free. Our assumption says that there is an irreducible representation $\bar{\pi}$ of $M(A\otimes\mathcal{K})$ with $\ker \bar{\pi} = A\otimes\mathcal{K}$ and $\tilde{t}\bar{\pi} \neq \bar{\pi}$ for $t \in G\setminus\{e\}$, which means that

$$\bigcap_{t\in G\setminus\{e\}} \{\, \bar{\pi} \in (M(A\otimes\mathcal{K})/(A\otimes\mathcal{K}))^\wedge \,|\, \tilde{t}\bar{\pi} \neq \bar{\pi} \,\} \neq \emptyset,$$

and hence $\tilde{\alpha}$ is topologically free by Remark 1.

REMARK 7. An action α satisfying the condition in the above theorem induces an outer action of G on a purely infinite simple C^*-algebra $M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$. In case α induces an inner action of G on $M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ so that $M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})\times_{\tilde{\alpha}} \cong M(A\otimes\mathcal{K})/(A\otimes\mathcal{K})\otimes C^*(G)$ for the group C^*-algebra C^* of G then the crossed product $A\times_\alpha G$ is purely infinite whenever $C^*(G)$ is simple since $C^*(G)$ is just a matrix algebra.

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