Bull. Korean Math. Soc. **31** (1994), No. 1, pp. 99–104

THE GROUP OF UNITS IN A LEFT ARTINIAN RING

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Let R be a left Artinian ring with identity 1 and let G be the group of units of R. It is shown that if G is finite, then R is finite. It is also shown that if 2.1 is a unit in R, then G is abelian if and only if R is commutative.

1. Introduction and basic definitions

An element a in R is said to be left quasi-regular if there exists $r \in R$ such that $r + a + ra = 0$. In this case, the element r is called a left quasi-inverse of a. A (right, left or two-sided) ideal I of R is said to be left quasi-regular if every element of I is left quasi-regular. Similarly, $a \in R$ is said to be right quasi-regular if there exists $r \in R$ such that $a + r + ar = 0$. Right quasi-inverse and right quasi-regular ideals are defined analogously. It is clear that if R has an identity 1, then a is left [resp. right] quasi-regular if and only if $1 + a$ is left [resp. right] invertible. The Jacobson radical J of R is defined by the left quasi-regular left ideal which contains every left quasi-regular left ideal of R. A ring R is said to be semisimple if its Jacobson radical J is zero. We note that R/J is semisimple.

In [2], Wedderburn-Artin have shown that if R is a semisimple left Artinian ring, then R is isomorphic to a direct sum of a finite number of simple rings. Hence we obtain the following:

THEOREM 1.1. *If R is a left Artinian ring with identity, then $R/J \cong \oplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \cdots, n$ and for some a positive integer n.*

Proof. See [2, Theorem 2.14, p.431 and Theorem 3.3, p.435].

Received November 10, 1992. Revised April 22, 1993.

2. Properties of R when G is finite and abelian

In this section, we shall denote G by the group of units of R and denote J by the Jacobson radical of R.

We begin with the following lemma:

LEMMA 2.1. *Let R be a ring, and let G^* be the group of units of R/J. Then $g \in G$ if and only if $g + J \in G^*$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that $g^* = g+J \in G^*$. Then there exists $h^* = h+J \in G^*$ such that $g^*h^* = h^*g^* = 1^*$ where 1^* is the identity of G^*. So $1-hg \in J$. By the definition of $J, 1 + J \subseteq G$ and so gh and $hg \in G$. It is clear that $g \in G$.

LEMMA 2.2. *Let R be a ring with identity. Then $a \in R$ is left quasi-regular if and only if $a + J \in R/J$ is left quasi-regular.*

Proof. It follows easily from Lemma 2.1.

THEOREM 2.3. *Let R be a left Artinian ring with identity 1. If G is finite group, then R is finite.*

Proof. By Theorem 1.1, $R/J \cong \oplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \cdots, n$ and for some a positive integer n. If G is finite, then by Lemma 2.1, G^*, the group of units of R/J, is also finite. Then D_i is finite for each $i = 1, 2, \cdots, n$. Indeed, suppose that D_i is infinite for some i. For simplicity of notation, we can assume $R/J = \oplus_{i=1}^{n} M_i(D_i)$. Consider a subset $G_i^* = \oplus_{i=1}^{n} H_i$ where $H_j = \{e_j\}, (e_j$ is the identity of $M_j(D_j)$ for $j \neq i$ and $H_i = \{(a_{st}) \in M_i(D_i) : a_{11} \in D_i \backslash \{0_i\}, a_{ss} = e_i \ (2 \leq s \leq n_i), a_{st} = 0_i \ (2 \leq s, t \leq n_i, s \neq t)$ and o_i (resp. e_i) is zero (resp. identity) of $D_i\}$. Then G_i^* is a subgroup of G^* and $|G_i^*| = |D_i \backslash \{0_i\}|$ is infinite, which contradicts to the fact that G^* is finite group. Hence D_i is finite for each $i = 1, 2, \cdots, n$, and so R/J is finite. Since $1 + J \subseteq G$ and G is finite, J is finite. Hence $|R| = |J| \cdot |R/J|$ is finite.

LEMMA 2.4. *Let R be a ring with identity and let G be the group of units of R. If G is abelian group and a and b are quasi-regular elements of R, then $ab = ba$. In particular, J is commutative.*

Proof. Since $1 + J \subset G$ and a and $b \in J$, $(1 + a)(1 + b) = (1 + b)(1 + a)$. Hence $ab = ba$. Since each element of J is quasi-regular, J is commutative.

REMARK. In Theorem 2.3, the condition that R has identity is necessary because p-Prüfer ring $Z(p^\infty)$ is infinite Artinian ring without identity which has no units.

LEMMA 2.5. *Let R be a left Artinian ring with identity. If G is abelian group, then $R/J \cong \oplus_{i=1}^{n} F_i$ where F_i is a field for each $i = 1, 2, \cdots, n$ and for some positive integer n.*

Proof. By Theorem 1.1, $R/J \cong \oplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \cdots, n$ and for some a positive integer n. First, we will show that each D_i is a field. Consider the subgroup $G_i^* = \oplus_{i=1}^{n} H_i$ of G^* given in the proof of Lemma 2.4. Since G^* is abelian, H_i is also abelian, and so D_i is abelian, that is, D_i is field. Let $D_i = F_i$. Next, we will show that $n_i = 1$ for each i. Assume that $n_i \geq 2$ for some i. Consider two elements $a = (a_{st})$ and $b = (b_{st})$ in $M_i(F_i)$ where if $s = t, a_{12} = a_{st} = e_i$, otherwise $a_{st} = 0_i$, and if $s = t, b_{21} = b_{st} = 1_i$, otherwise $b_{st} = 0_i$. By the simple calculation, we have $(1, 1)$-entry of $ab = 2 \neq 1 = (1, 1)$-entry of ba. Thus the group of units in $M_i(F_i)$ is not abelian, and so G^* is not abelian group, which is a contradiction. Hence we have the result.

Let R be a left Artinian ring with identity such that G is abelian group. By Lemma 2.5, $R/J \cong \oplus_{i=1}^{n} F_i$ where F_i is field for each $i (1 \leq i \leq n)$ and for some positive integer n. For simplicity of notation, we can assume that $R/J \cong \oplus_{i=1}^{n} F_i$. Let $\phi : R \to R/J$ denote the canonical epimorphism and for ecah i, let $R_i = \phi^{-1}(\oplus_{i=1}^{n} H_i)$ where $H_j = \{0_j\}$ (0_j is additive identity of F_j) for $j \neq i$ and $H_i = F_i$. Let $\phi_i = \phi|_{R_i}$. Then Ker $\phi_i = \{a \in R_i : \Pi_i(\phi_i(a)) = 0_i\}$ where Π_i is the projection of $\oplus F_j$ to F_1. Note that Ker $\phi_i = J$ for each $i = 1, 2, \cdots, n$ and each R_i is an ideal of R. If 1_i is the identity of F_i, let 1_i^* denote the identity of $\phi_i = \oplus_{i=1}^{n} H_j$, that is, $1_i^* = \oplus_{i=1}^{n} a_j$ where $a_j = 0_j$ for $j \neq i$ and $a_i = 1_i$. Observe that $\phi_i^{-1}(\{1_i^*\})$ is contained in the center of R_i if and only if $\phi_i^{-1}(\{-1_i^*\})$ is contained in the center of R_i.

LEMMA 2.6. *Let $\phi : R \to R'$ be a ring epimorphism. If A and B are subsets of R', then $\phi^{-1}(A+B) = \phi^{-1}(A) + \phi^{-1}(B)$.*

Proof. If $x \in \phi^{-1}(A+B)$, then $\phi(x) = a + b \in A + B$. Since ϕ is onto, there exist $a* \in A$ and $b* \in B$ such that $\phi(a*) = a$ and $\phi(b*) = b$. So $\phi(x) = a + b = \phi(a*) + \phi(b*) = \phi(a* + b*) \in \phi(\phi^{-1}(A) + \phi^{-1}(B))$. Hence $x \in \phi^{-1}(A) + \phi^{-1}(B)$.

If $x \in \phi^{-1}(A) + \phi^{-1}(B)$, then $x = a* + b*$ where $a* \in \phi^{-1}(A)$ and $b* \in \phi^{-1}(B)$. So $\phi(x) = \phi(a* + b*) = \phi(a*) + \phi(b*) \in A + B$. Hence $x \in \phi^{-1}(A+B)$.

LEMMA 2.7. *If R is a left Artinian ring with identity, then $R = R_1 + R_2 + \cdots + R_n$ where $R_i = \phi^{-1}(\oplus_{i=1}^{n} H_i)$ with $H_j = \{0_j\}$ (0_j is additive identity of F_j) for $j \neq i$ and $H_i = F_i$.*

Proof. Let $F_i^* = \oplus_{i=1}^{n} H_i$ for each i. Then $\oplus_{i=1}^{n} F_i = F_1^* + F_2^* + \cdots + F_n^*$. Hence $R = \phi^{-1} \circ \phi(R) = \phi^{-1}(R/J) = \phi^{-1}(\oplus F_i) = \phi^{-1}(F_1^* + F_2^* + \cdots + F_n^*) = \phi^{-1}(F_1^*) + \phi^{-1}(F_2^*) + \cdots + \phi^{-1}(F_n^*) = R_1 + R_2 + \cdots + R_n$ by Lemma 2.6.

LEMMA 2.8. *Let R be a ring with identity such that G is abelian group and $R/J = \oplus_{i=1}^{n} F_i$ where each F_i is field. If $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ (= center of R_i), then R_i is commutative.*

Proof. Since R_i is an ideal of R, if $a \in R_i$, then a is quasi-regular in R_i if and only if a is quasi-regular in R. Hence by Lemma 2.2, if $a \in R_i$, then a is quasi-regular in R_i if and only if $\phi(a)$ is quasi-regular in R/J, that is , $\phi_i(a)$ is quasi-regular in $F_1^* = \oplus_{i=1}^{n} H_j$ where $H_j = \{0_j\}$ for $j \neq i$ and $H_i = F_i$. Hence for $a \in R_i$, a is quasi-regular if and only if $\Pi_i(\phi_i(a)) + 1_i \neq 0_i$.

Now let $a, b \in R_i$. If a and b are quasi-regular, then $ab = ba$ by Lemma 2.4. If a is not quasi-regular, then $\Pi_i(\phi_i(a)) + 1_i = 0_i$, that is, $a \in \phi_i^{-1}(\{-1_i^*\})$. Thus a is in the center of R_i and so $ab = ba$. Similarly, if b is not quasi-regular, then $ab = ba$.

LEMMA 2.9. *Let R be a ring with identity such that G is abelian group and $R/J = \oplus_{i=1}^{n} F_i$ where each F_i is field. If $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ (= center of R_i) for all $i = 1, 2, \cdots, n$, then R is commutative.*

Proof. Let $a \in R_i$ and $b \in R_j$ for $i \neq j$ ($1 \leq i,\ j \leq n$). By Lemma 2.7, it suffices to show that $ab = ba$. By Lemma 2.4, we may assume that both a and b are not quasi-regular. Without loss of generality, we may assume that a is not quasi-regular. Then $\Pi_i(\phi_i(a)) = -1_i$. Since $ab = ba$ if and only if $(-a)b = b(-a)$, we may assume that $\Pi_i(\phi_i(a)) = 1_i$. Now $ab, ba \in R_i \cap R_j$ since R_i and R_j are ideals of R. But for $i \neq j$, $R_i \cap R_j = J$. So $ab, ba \in J$. Since $J \subseteq Z(R_i)$ for each i, ab and ba are in $Z(R_i)$ for each i. Hence $a(ab) = (ab)a = a(ba) = (ba)a$, that is , $a^2b = ba^2$. Since $\Pi_i(\phi_i(a^2-a)) = 0_i$, $a^2 - a \in J$. So $(a^2-a) = b(a^2-a)$. Hence $-ab = -ba$, that is, $ab = ba$.

LEMMA 2.10. *Let R be a ring with identity such that G is abelian group and $R/J = \oplus_{i=1}^n F_i$ where each F_i is field. If char $(F_i) \neq 2$ for some i, then $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ (= center of R_i).*

Proof. Since char $(F_i) \neq 2$ for some i, $1_i \neq -1_i$. For any $u_i \in F_i \backslash \{0_i, -1_i\}$, there exists $w)i \in F_i \backslash \{0_i, -1_i\}$ such that $u_i \cdot w_i = 1_i$. So $w_i + 1_i \neq 0_i$ and $u_i + 1_i \neq 0_i$, and hence u_i and w_i are quasi-regular elements of F_i. Let $u = (0_1, \cdots, 0_{i-1}, u_i, 0_{i+1}, \cdots, 0_n)$ and $w = (0-1, \cdots, 0_{i-1}, w_i, 0_{i+1}, \cdots, 0_n)$. Then u and w are quasi-regular in $\oplus H_i$ where $H_j = \{0_j\}$ for $j \neq i$ and $H_i = F_i$. Since ϕ_i is onto, there exist a, b and $e \in R_i$ such that $\phi_i(a) = u$, $\phi_i(b) = w$ and $\phi_i(e) = 1_i$. Then $\Pi_i(\phi_i(e-ab) = \Pi_i(1-uw) = 0_i$, so $e - ab \in$ Ker $\phi_i = J$. Note that a and b are quasi-regular in R if and only if $\varphi(a)$ and $\phi(b)$ are quasi-regular in R/J. Let x be arbitrary element of R_i. If x is quasi-regular, then by Lemma 2.4, $x(e-ab) = (e-ab)x$ since $e - ab \in J$. Hence $xe - xab = ex - abx$. Since a and b are qasi-regular, $xab = abx$. Thus $xe = ex$. If x is not quasi-regular, then $\Pi_i(\phi_i(x)) = -1_i = \Pi_i(\phi_i(-e))$. So $x + e \in$ Ker $\phi_i = J$. Thus $x + e = j$ for some $j \in J$. Since j is quasi-regular in $R_i, ej = je$. So $xe = (j-e)e = je - e^2 = ej - e^2 = e(j-e) = ex$. Thus $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$.

THEOREM 2.11. *Let R be a left Artinian ring with identity 1 such that $2 = 2 \cdot 1$ is a unit in R. Then G is abelian if and only if R is commutative.*

Proof. ($\Leftarrow$) Clear.
($\Rightarrow$) Suppose that G is abelian. Then $R/J \cong \oplus_{i=1}^n F_i$ where F_i is a field for each $i = 1, 2, \cdots, n$ and for some positive integer n. For simplicity of notation, we can assume that $R/J \cong \oplus_{i=1}^n F_i$. Since 2 is

unit in R, then $2+J$ is unit in R/J by Lemma 2.1. So $\mathrm{char}(F_i) \neq 2$ for each $i = 1, 2, \cdots, n$. Therefore, the theorem follows from Lemma 2.6, Lemma 2.9 and Lemma 2.10.

REMARK. In Theorem 2.11, the condition that 2 is a unit in R is essential, since the ring R of upper triangular 2×2 matrices over Z_2 is not commutative but the group of units of R is abelian.

COROLLARY 2.12. *Let R be a left Artinian ring with identity* 1 *such that* $2 = 2 \cdot 1$ *is a unit in R. If G is cyclic, then R is a finite commutative ring.*

Proof. If G is cyclic, G is abelian. So by Theorem 2.11 R is commutative. Moreover, if G is cyclic, then R is finite. [See [3]]

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