Bull. Korean Math. Soc. **31** (1994), No. 1, pp. 15–23

ON THE ADJOINT LINEAR SYSTEM

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§ 0. Introduction.

Throughout this paper, we are working on the complex number field **C**.

The aim of this paper is to explain the applications of Theorem 2 in §1. In the surface theory, the adjoint linear system has played important roles and many tools have been developed to understand it. In the cases of higher dimensional varieties, we don't have any useful tools so far. Theorem 2 implies that it is enough to compute the dimension of the adjoint linear system to check the birationality. We can compute, somehow, the dimension of the adjoint linear system. For example, we can get an information about $h^0(X, \mathcal{O}_X(K_X + D))$ from Euler characteristic of $|K_X + D|$ and some vanishing theorems.

We are going to show the applications of Theorem 2 to smooth threefolds and smooth fourfold, specially, of general type with a nef canonical divisor, smooth Fano variety, and Calabi-Yau manifold. Our main results are Theorem A and Theorem B. Most of birationality problems in Theorem A and Theorem B have been studied. (See Ando [1] and Matsuki [4] for the detail matters.) But Theorem 2 gives short and easy proofs in the cases of dimension 3 and improves the previously known results in the cases of dimension 4.

§ 1. Main

Let X be a smooth projective variety. We denote a linear equivalence by $\sim$. Denote by $\mathrm{Div}(X)$ a free abelian group generated by the divisors on X. Denote the canonical divisor of X by K_X. Then we say that $D \in$

Received August 7, 1992.
This paper is supported by SNU–GARC.

$\mathrm{Div}(X)$ is *nef* if $D \cdot C \geq 0$ for any curve C on X, and *big* if $\kappa(D, X) = \mathit{dim}\ X$, where $\kappa(D, X)$ is the Kodaira dimension of D on X. For $D \in \mathrm{Div}(X)$, $\Phi_{|D|}$ denotes the rational map associated with the complete linear system $|D|$ if $h^0(X, \mathcal{O}_X(D)) \neq 0$. Let's denote $h^0(X, \mathcal{O}_X(nD))$ by $p_n(D)$.

THEOREM 1. *(Kawamata-Viehweg vanishing Theorem) Let X be a nonsingular projective variety and $D \in \mathrm{Div}(X)$. If D is nef and big, then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

For a proof, see Kawamata [3].

LEMMA 1. *Let X be a smooth projective threefold, and $D \in \mathrm{Div}(X)$. Then we have the following:*

(1) $\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$, *where c_2 is the second Chern class of X. Moreover $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24$.*

(2) $K_X \cdot D^2$ *is even.*

Proof. (1) is the Riemann-Roch Theorem.

(2) comes from the following:

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbf{Z}. \quad \square$$

LEMMA 2. *Let X be a smooth threefold with a canonical divisor K_X. Let $D \in \mathrm{Div}(X)$.*

(1) *When K_X is nef and big,* $p_n(K_X) = \dfrac{n(n-1)(2n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X)$ *for $n \geq 2$.*

(2) *When $-K_X$ is ample,* $p_n(-K_X) = \dfrac{n(n+1)(2n+1)}{12}(-K_X^3) + (2n+1)$ *for $n \geq 1$.*

(3) *When $K_X \sim 0$, and D is nef and big,* $p_n(D) = \dfrac{n^3 D^3}{6} + \dfrac{nD \cdot c_2}{12}$ *for $n \geq 1$.*

Proof. Suppose that $L \in \mathrm{Div}(X)$ is nef and big.

$$\begin{aligned}\chi(\mathcal{O}_X(K_X+L)) = \; & h^0(X, \mathcal{O}_X(K_X+L)) - h^1(X, \mathcal{O}_X(K_X+L)) \\ + \; & h^2(X, \mathcal{O}_X(K_X+L)) - h^3(X, \mathcal{O}_X(K_X+L)).\end{aligned}$$

Since L is nef and big, $h^i(X, \mathcal{O}_X(K_X+L)) = 0$ for $i > 0$ by Theorem 1. Thus
$\chi(\mathcal{O}_X(K_X+L)) = h^0(X, \mathcal{O}_X(K_X+L))$.

For (1), take $L = (n-1)K_X$.

$$\begin{aligned}p_n(K_X) =& h^0(X, \mathcal{O}_X(K_X+(n-1)K_X)) \\ =& \chi(\mathcal{O}_X(K_X+(n-1)K_X)).\end{aligned}$$

Then our claim follows from (1) of Lemma 1.

For (2), take $L = (n+1)(-K_X)$.

$$\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^3(X, \mathcal{O}_X).$$

For $i > 0$, $h^i(X, \mathcal{O}_X) = h^{3-i}(X, \mathcal{O}_X(K_X)) = 0$ since $-K_X$ is ample. So $\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$.

$$\begin{aligned}p_n(K_X) =& h^0(X, \mathcal{O}_X(K_X+(n+1)(-K_X))) \\ =& \chi(\mathcal{O}_X(K_X+(n+1)(-K_X))).\end{aligned}$$

And apply (1) of Lemma 1.

For (3), take $L = nD$. Since $K_X \sim 0$, $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24 = 0$. We will get our claim from (1) of Lemma 1. □

LEMMA 3. *Let X be a smooth threefold with a canonical divisor K_X. Let $D \in \mathrm{Div}(X)$.*

(1) *When K_X is nef and big, $p_n(K_X) \geq 4$ for $n \geq 2$.*
(2) *When $-K_X$ is ample, $p_n(-K_X) \geq 4$ for $n \geq 1$.*
(3) *When $K_X \sim 0$, and D is nef and big, $p_n(D) \geq 2$ for $n \geq 2$.*

Proof. Let $L \in \mathrm{Div}(X)$. If K_X and L both are nef, then $L \cdot (3c_2 - c_1^2) \geq 0$ by the pseudo-effectivity of $3c_2 - c_1^2$ (See Miyaoka [5].)

For (1), take $L = K_X$. It follows that $\chi(\mathcal{O}_X) < 0$ from (1) of Lemma 1. Since K_X^3 is a positive even integer, and $\chi(\mathcal{O}_X) < 0$,

$$p_n(K_X) \geq \frac{n(n-1)(2n-1)}{6} + (2n-1) \geq 4 \qquad \text{for } n \geq 2.$$

For (2), $-K_X^3$ is a positive even integer since $-K_X$ is ample.

$$p_n(-K_X) \geq \frac{n(n+1)(2n+1)}{6} + (2n+1) \geq 4 \qquad \text{for } n \geq 1.$$

For (3), take $L = nD$. Since $K_X \sim 0$, and D is nef, we have $D \cdot c_2 \geq 0$. Thus,

$$p_n(D) \geq \frac{n^3 D^3}{6} \geq \frac{4}{3} \qquad \text{for } n \geq 2.$$

Hence $p_n(D) \geq 2$ for $n \geq 2$. □

THEOREM 2. *Let X be a smooth projective threefold and let D be a nef and big divisor on X. Assume that $h^0(X, \mathcal{O}_X(mD)) \geq 2$ for some positive integer m. Then $\Phi_{|K_X+nD|}$ is birational for a positive integer $n \geq m+4$ such that $h^0(X, \mathcal{O}_X((n-m)D)) \geq 1$.*

For a proof, see Shin [6].

THEOREM A. *Let X be a smooth projective threefold with a canonical divisor K_X and let D be a nef and big divisor on X.*

(1) *When K_X is nef and big, $\Phi_{|nK_X|}$ is birational for $n \geq 7$. (cf. See Matsuki [4].)*
(2) *When $-K_X$ is ample, $\Phi_{|-nK_X|}$ is birational for $n \geq 4$.*
(3) *When $K_X \sim 0$, $\Phi_{|nD|}$ is birational for $n \geq 6$.*

Proof. We are going to apply Theorem 2 to each case. So, first of all, we have to choose the number "m" in the Theorem 2 as small as we can.

For (1), take $m = 2$. For an integer $n \geq 7$, $\Phi_{|nK_X|} = \Phi_{|K_X+(n-1)K_X|}$ and $n-1 \geq m+4$. $h^0(X, \mathcal{O}_X((n-1)-m)K_X)) \geq 4$ by Lemma 3 since

$n-1-m \geq 2$. Hence Theorem 2 implies that $\Phi_{|nK_X|}$ is birational for $n \geq 7$.

For (2), take $m = 1$. For an integer $n \geq 4$, by similar way, we can show that n satisfies all the conditions in Theorem 2. Hence $\Phi_{|nK_X|}$ is birational for $n \geq 4$.

For (3), take $m = 2$. Since $K_X \sim 0$, $\Phi_{|K_X+nD|} = \Phi_{|nD|}$. For an integer $n \geq 6$, n satisfies all the conditions in Theorem 2. Hence $\Phi_{|nK_X|}$ is birational for $n \geq 6$. □

REMARK. In the case (3) of the Theorem A, a little modification of the proof of Theorem 2 can improve our result to $n \geq 5$.

LEMMA 4. *Let X be a smooth projective fourfold and let $L \in Div(X)$.*

(1) $\chi(\mathcal{O}_X(L)) = \frac{1}{24}[L^4 - 2K_X \cdot L^3 + (K_X^2 + c_2) \cdot L^2 - c_2 \cdot K_X \cdot L + 24\chi(\mathcal{O}_X)]$.

(2) *When $L = nK_X$, $p_n(K_X) = \frac{1}{24}K_X^4[n^2(n-1)^2 + an(n-1) + b]$ for $n \geq 2$, where $a = \frac{c_2 \cdot K_X^2}{K_X^4}$ and $b = 24\chi(\mathcal{O}_X)/K_X^4$. Moreover, $a \geq 1/3$.*

(3) *When $L = -nK_X$, $p_n(-K_X) = \frac{1}{24}[n^2(n+1)^2K_X^4 + n(n+1)c_2 \cdot K_X^2 + 24\chi(\mathcal{O}_X)]$ for $n \geq 0$.*

(4) *When $K_X \sim 0$ and $L = nD$, $p_n(D) = \frac{1}{24}[n^4D^4 + n^2c_2 \cdot D^2 + 24\chi(\mathcal{O}_X)]$ for $n \geq 1$.*

Proof. (1) is from Riemann-Roch theorem. $a \geq 1/3$ by Miyaoka [5]. (2), (3) and (4) come from (1) and Kawamata-Viehweg theorem. □

LEMMA 5. *Let X be a smooth fourfold of general type with K_X nef.*

(1) *$p_n(K_X) \geq 2$ for $n \geq 3$ and $p_n(K_X) \geq 3$ for $n \geq 4$.*

(2) *$dim\Phi_{|nK_X|}(X) \geq 2$ for $n \geq 4$.*

(3) *$p_n(K_X) - p_{n-4}(K_X) \geq 2$ for $n \geq 3$.*

Proof. For (1), since $p_2(K_X) \geq 0$, we have $2a + b \geq -4$. Thus $p_n(K_X) \geq \frac{K_X^4}{24}\{n^2(n-1)^2 + 1/3(n-2)(n+1) - 4\}$ for $n \geq 2$. Hence $p_n(K_X) \geq 2$ for $n \geq 3$ and $p_n(K_X) \geq 3$ for $n \geq 4$.

(2) comes from the fact that

$$dim\Phi_{|nK_X|}(X) > r \text{ if } h^0(X, \mathcal{O}_X(nK_X)) > n^r K_X^4 + r.$$

(See T. Ando [1].)

For (3),

$$\begin{aligned} & p_n(K_X) - p_{n-4}(K_X) \\ =& \frac{K_X^4}{24}\{16n^3 - 120n^2 + 360n - 400 + a(8n - 20)\} \\ \geq& \frac{K_X^4}{24}\{16n^3 - 120n^2 + 1088n/3 - 1220/3\} \\ \geq& 2 \text{ for } n \geq 6. \end{aligned}$$

When $n = 5$, $p_5(K_X) - p_g \geq 2$ regardless of p_g.

When $n = 4$, $p_4(K_X) - 1 \geq 2$ by (2).

When $n = 3$, $p_3(K_X) - h^0(X, \mathcal{O}_X(-K_X)) \geq 2$, since $h^0(X, \mathcal{O}_X(-K_X)) = 0$.

Therefore $p_n(K_X) - p_{n-4}(K_X) \geq 2$ for $n \geq 3$. □

LEMMA 6. *Let X be a smooth fourfold with $-K_X$ ample.*

(1) *$p_n(-K_X) \geq 3$ for $n \geq 4$.*

(2) *$dim\Phi_{|-4K_X|}(X) \geq 2$.*

(3) *$p_n(K_X) - p_{n-4}(K_X) \geq 3$ for $n \geq 4$.*

Proof. $p_n(-K_X) = \chi(\mathcal{O}_X(-nK_X))$ and $\chi(\mathcal{O}_X) = 1$ since $-K_X$ is ample.

For (1), $p_1(-K_X) \geq 0$ implies that $\frac{1}{24}[4K_X^4 + 2c_2 \cdot K_X^2] + 1 \geq 0$. Thus $c_2 \cdot K_X^2 \geq -2K_X^4 - 12$.

$$\begin{aligned} p_n(-K_X) &\geq \frac{1}{24}[n^2(n+1)^2K_X^4 + n(n+1)(-2K_X^4 - 12)] + 1 \\ &= \frac{1}{24}n(n+1)[(n^2 + n - 2)K_X^4 - 12] + 1 \\ &\geq 3 \text{ for } n \geq 4. \end{aligned}$$

For (2), from (1), $p_n(-K_X) \geq \frac{1}{24}n(n+1)[(n^2+n-2)K_X^4 - 12] + 1$ for $n \geq 4$.

$$\begin{aligned} &\{\frac{1}{24}n(n+1)[(n^2+n-2)K_X^4 - 12] + 1\} - \{nK_X^4 + 1\} \\ =&\frac{n}{24}[(n^3 + 2n^2 - n - 26)K_X^4 - 12(n+1)] > 0 \text{ for } n \geq 4. \end{aligned}$$

Hence $dim\Phi_{|-4K_X|}(X) \geq 2$.

For (3), $p_n(K_X) - p_{n-4}(K_X) \geq \frac{1}{24}[(16n^3 - 72n^2 + 152n - 120)K_X^4 - 96n + 144] \geq 2$ for $n \geq 4$. □

REMARK. When $K_X^4 = 1$, $c_2 \cdot K_X^2$ should be of the form $12k + 10$, where k is an integer. Hence $p_n(-K_X) \geq 3$ for $n \geq 3$ except the case $(K_X^4, c_2 \cdot K_X^2) = (1, -14)$.

LEMMA 7. *Let X be a smooth projective fourfold with $K_X \sim 0$ and D be nef and big.*

(1) $p_n(D) \geq 3$ *for* $n \geq 3$.
(2) $dim\Phi_{|4D|}(X) \geq 2$.
(3) $p_n(D) - p_{n-4}(D) \geq 2$ *for* $n \geq 3$.

Proof. $p_1(D) \geq 0$, so $\chi(\mathcal{O}_X) \geq -\frac{D^4 + c_2 \cdot D^2}{24}$. Since $c_2 \cdot D^2 \geq 0$ by Miyaoka [5], we have

$$\begin{aligned} p_n(D) &\geq \frac{1}{24}[n^4 D^4 + n^2 c_2 \cdot D^2 - D^4 - c_2 \cdot D^2] \\ &\geq \frac{n^4 - 1}{24} D^4 \text{ for } n \geq 1. \end{aligned}$$

So (1), (2) and (3) come from similar arguments in the Lemma 5. □

THEOREM B. *Let X be a smooth projective fourfold with a canonical divisor K_X and let $D \in Div(X)$ be nef and big.*

(1) *When K_X is nef and big, then $\Phi_{|nK_X|}$ is birational for $n \geq 11$.*
(2) *When $-K_X$ is ample, then $\Phi_{|-nK_X|}$ is birational for $n \geq 11$.*
(3) *When D is nef and big, then $\Phi_{|nD|}$ is birational for $n \geq 11$.*

Proof. (1) Resolve the base locus of $\Phi_{|4K_X|}$. Then we have a morphism $f : X' \to X$ such that

$$\text{(i) } |4f^*K_X| = |S_4| + F$$
$$\text{(ii) } |S_4| \text{ is base-point free,}$$

where $|S_4|$ is the moving part of $|4f^*K_X|$ and F is the fixed part of $|4f^*K_X|$. Then a general member of $|S_4|$, say S_4, is a smooth threefold since $dim\Phi_{|4K_X|}(X) \geq 2$ and $|S_4|$ is base-ponit free.

Consider the following exact sequence:

$$0 \to \mathcal{O}_{X'}(K_{X'} + (n-5)f^*K_X) \to \mathcal{O}_{X'}(K_{X'} + S_4 + (n-5)f^*K_X)$$
$$\to \mathcal{O}_{S_4}(K_{S_4} + (n-5)R) \to 0,$$

where $R = f^*K_{X|S_4}$. Hence if $\Phi_{|K_{S_4}+(n-5)R|}$ is birational, then so is $\Phi_{|nK_X|}$. To apply Theorem 2, we need to compute $h^0(S_4, \mathcal{O}_{S_4}((n-5)R))$.

Consider the following exact sequence: for a positive integer k,

$$0 \to \mathcal{O}_{X'}(kf^*K_X - S_4) \to \mathcal{O}_{X'}(kf^*K_X) \to \mathcal{O}_{S_4}(kR) \to 0.$$

So

$$h^0(S_4, \mathcal{O}_{S_4}(3R)) \geq h^0(X', \mathcal{O}_{X'}(3f^*K_X)) - h^0(X', \mathcal{O}_{X'}(3f^*K_X - S_4))$$
$$\geq 2.$$

And $h^0(S_4, \mathcal{O}_{S_4}(kR)) \geq 1$ for $k \geq 3$ because of (1) in Lemma 5.

If we take $m = 3$ in the Theorem 2, $\Phi_{|K_{S_4}+(n-5)R|}$ is birational for $n \geq 11$.

Proofs of (2) and (3) are very similar to (1). □

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