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REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM IN TERMS OF RICCI TENSORS

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§1. Introduction.

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0, c = 0$ or $c < 0$.

In his study of real hypersurfaces of $P_n\mathbb{C}$, Takagi [9] classified all homogeneous real hypersurfaces and Cecil and Ryan [3] showed also that they are realized as the tubes of constant radius over Kähler submanifolds if the structure vector field ξ is principal. Real hypersurfaces of $H_n\mathbb{C}$ have also investigated by Berndt [2], Montiel [5], Montiel and Romero [6] and so on, and Berndt [2] classified all homogeneous real hypersurfaces of $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given.

Now, let M be a real hypersurface of $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler metric and the almost complex structure of $M_n(c)$. We denote by A the shape operator in the direction of the unit normal on M. Then Okumura [7] and Montiel and Romero [6] proved the following

THEOREM A. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 2$. If it satisfies*

$$A\phi - \phi A = 0, \tag{1.1}$$

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then M is locally a tube of radius r over one of the following Kähler submanifolds:

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where* $0 < r < \pi/2$,
(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$, where* $0 < r < \pi/2$.

THEOREM B. *Let M be a real hypersurface of $H_n\mathbb{C}, n \geqq 2$. If it satisfies (1.1), then M is locally one of the following hypersurfaces:*

(A_0) *a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,*
(A_1) *a tube of a totally geodesic hyperplane $H_{n-1}\mathbb{C}$,*
(A_2) *a tube of a totally geodesic $H_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$.*

Such real hypersurfaces in Theorems A and B are said to be of *type A*. Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution*. The second fundamental form is said to be *η-parallel* if the shape operator A satisfies $g(\nabla_X A(Y), Z) = 0$ for any vector fields X, Y and Z in T_0, where $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X. Then the following is recently proved by Ahn, Lee and Suh [1].

THEOREM C. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. Assume that the structure vector field ξ is not principal. Then it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{1.2}$$

for any vector fields X and Y in T_0 and the second fundamental form is η-parallel if and only if M is locally a ruled real hypersurface.

On the other hand, Kimura and Maeda [4] and Suh [8] classified real hypersurfaces in $M_n(c), c \neq 0, n \geqq 2$ which satisfy the conditions that ξ is principal and the Ricci tensor is η-parallel.

The purpose of this article is to characterize real hypersurfaces of type A in $M_n(c), c \neq 0$ for the Ricci tensor in spite of the shape operator and to prove the following

THEOREM. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 2$. If it satisfies (1.2) and if the Ricci tensor S is η-parallel, then M is of type A.*

§2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $(M_n(c), \bar{g})$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor $\bar{g}$ on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained :

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.2}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Now, we here calculate the covariant derivative of the Ricci tensor S. Since the Ricci tensor S is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2$$

for the identity transformation I and the trace h of A,

$$\begin{aligned}\nabla_X S(Y) = &-\frac{3}{4}cg(\phi AX, Y)\xi + dh(X)AY \\ &+ h\nabla_X A(Y) - \nabla_X A(AY) - A\nabla_X A(Y),\end{aligned}$$

from which it turns out to be

$$\begin{aligned}g(\nabla_X S(Y), Z) =& dh(X)g(AY, Z) + hg(\nabla_X A(Y), Z) \\ &- g(\nabla_X A(Y), AZ) - g(\nabla_X A(Z), AY)\end{aligned} \tag{2.4}$$

for any vector fields X, Y and Z in T_0.

The Ricci tensor S is said to be *η-parallel* if it satisfies $g(\nabla_X S(Y), Z) = 0$ for any vector fields X, Y and Z in T_0. It follows that if ξ is principal and if the second fundamental form is η-parallel, then S is also η-parallel by (2.4). (See [4] and [8].)

§3. Proof of Theorem.

Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 2$ and assume that the Ricci tensor S is η-parallel. Namely, we assume that

$$g(\nabla_X S(Y), Z) = 0, \quad X, Y, Z \in T_0. \tag{3.1}$$

Suppose that the structure vector field ξ is not principal. We put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. Then it means that the function β does not vanish identically on M. We denote by M_0 the non-empty open subset of M consisting of points x at which $\beta(x) \neq 0$. Hereafter unless otherwise stated, we shall discuss on the subset M_0 of M. We here assume the following condition:

$$g((A\phi - \phi A)X, Y) = 0, \quad X, Y \in T_0. \tag{1.2}$$

By the above assumption, it turns out to be

$$(A\phi - \phi A)X = -\beta g(X, \phi U)\xi, \quad X \in T_0. \tag{3.2}$$

Making use of this property, we have

$$\begin{aligned} &g(\nabla_X A(Y), A\phi Z) + g(\nabla_X A(\phi Y), AZ) \\ &\quad = g(\nabla_X A(Y), \phi AZ) - \beta g(Z, \phi U) g(\nabla_X A(Y), \xi) \\ &\qquad + g(\nabla_X A((AZ)_0), \phi Y) + \beta g(Z, U) g(\nabla_X A(\xi), \phi Y) \end{aligned} \tag{3.3}$$

for any vector fields X, Y and Z in T_0, where $(AZ)_0$ denotes the T_0-component of AZ. Furthermore, by the form $A\xi = \alpha\xi + \beta U$, we get

$$\begin{aligned} &g(\nabla_X A(Y), \phi Z) + g(\nabla_X A(Z), \phi Y) \\ &\quad = \beta\{g(Y, U) g(AX, Z) + g(Z, U) g(AX, Y) \\ &\qquad - g(Y, \phi U) g(\phi AX, Z) - g(Z, \phi U) g(\phi AX, Y)\} \end{aligned} \tag{3.4}$$

for any vector fields X, Y and Z in T_0. Hence the equation (3.3) is reformed as

(3.5)
$$\begin{aligned} &g(\nabla_X A(Y), A\phi Z) + g(\nabla_X A(\phi Y), AZ) \\ &\quad = \beta\{g(Y, U) g(AX, AZ) + g(AZ, U) g(AX, Y) \\ &\qquad - g(Y, \phi U) g(\phi AX, AZ) - g(AZ, \phi U) g(\phi AX, Y) \\ &\qquad - \beta^2 g(X, U) g(Y, U) g(Z, U) + g(Z, U) g(\nabla_X A(\phi Y), \xi) \\ &\qquad - g(Z, \phi U) g(\nabla_X A(Y), \xi)\}. \end{aligned}$$

On the other hand, we have by (1.2) and (2.4)

$$\begin{aligned}
&g(\nabla_X S(Y), \phi Z) + g(\nabla_X S(Z), \phi Y) \\
&\quad = h\{g(\nabla_X A(Y), \phi Z) + g(\nabla_X A(Z), \phi Y)\} \\
&\qquad - g(\nabla_X A(Y), A\phi Z) - g(\nabla_X A(\phi Y), AZ) \\
&\qquad - g(\nabla_X A(Z), A\phi Y) - g(\nabla_X A(\phi Z), AY).
\end{aligned}$$

for any vector fields X, Y and Z in T_0. From (3.4), (3.5) and the above equation, we obtain

$$\begin{aligned}
&g(\nabla_X S(Y), \phi Z) + g(\nabla_X S(Z), \phi Y) \\
&\quad = \beta[h\{g(Y,U)g(AX,Z) + g(Z,U)g(AX,Y) \\
&\qquad - g(Y,\phi U)g(\phi AX, Z) - g(Z,\phi U)g(\phi AX, Y)\} \\
&\qquad -g(Y,U)g(AX,AZ) - g(AZ,U)g(AX,Y) \\
&\qquad +g(Y,\phi U)g(\phi AX, AZ) + g(AZ,\phi U)g(\phi AX, Y) \\
&\qquad -g(Z,U)g(AX,AY) - g(AY,U)g(AX,Z) \\
&\qquad +g(Z,\phi U)g(\phi AX,AY) + g(AY,\phi U)g(\phi AX, Z) \\
&\qquad -g(Y,U)g(\nabla_X A(\phi Z),\xi) - g(Z,U)g(\nabla_X A(\phi Y),\xi) \\
&\qquad +g(Y,\phi U)g(\nabla_X A(Z),\xi) + g(Z,\phi U)g(\nabla_X A(Y),\xi) \\
&\qquad +2\beta^2 g(X,U)g(Y,U)g(Z,U)]
\end{aligned} \tag{3.6}$$

for any vector fields X, Y and Z in T_0.

Next, taking account of (1.2) and the first equation of (2.1), we have

$$\begin{aligned}
g(\nabla_X A(Y), \xi) = {} & \alpha g(\phi AX, Y) - g(\phi AX, AY) \\
& + d\beta(X)g(Y,U) + \beta g(\nabla_X U, Y),
\end{aligned} \tag{3.7}$$

and hence, by the property of the structure tensor ϕ , we get also

$$\begin{aligned}
g(\nabla_X A(\phi Y), \xi) = {} & \alpha g(AX,Y) - g(AX,AY) + \beta^2 g(X,U)g(Y,U) \\
& - d\beta(X)g(Y,\phi U) + \beta g(\nabla_X U, \phi Y)
\end{aligned}$$

for any vector fields X and Y in T_0. By substituting the above two equations into (3.6) and by the assumption (3.1), it follows that the shape operator A satisfies

$$
\begin{aligned}
& (h-\alpha)\{g(Y,U)g(AX,Z)+g(Z,U)g(AX,Y) \\
& \qquad -g(Y,\phi U)g(\phi AX,Z)-g(Z,\phi U)g(\phi AX,Y)\} \\
& -g(AY,U)g(AX,Z)-g(AZ,U)g(AX,Y) \\
& +g(AY,\phi U)g(\phi AX,Z)+g(AZ,\phi U)g(\phi AX,Y) \\
& +2d\beta(X)\{g(Y,U)g(Z,\phi U)+g(Z,U)g(Y,\phi U)\} \\
& -\beta\{g(Y,U)g(\nabla_X U,\phi Z)+g(Z,U)g(\nabla_X U,\phi Y) \\
& \qquad -g(Y,\phi U)g(\nabla_X U,Z)-g(Z,\phi U)g(\nabla_X U,Y)\} \\
= \ & 0
\end{aligned} \tag{3.8}
$$

for any vector fields X, Y and Z in T_0.

If n=2, then we can put

$$AU=\beta\xi+\gamma U+\delta\phi U, \quad A\phi U=\delta U+\varepsilon\phi U,$$

where γ,δ and ε are smooth functions on M_0. Since $A\phi U=\phi AU$ by (3.2), we get

$$\delta=0, \quad \gamma=\varepsilon. \tag{3.9}$$

Next, we suppose that $n \geqq 3$ and let T_1 be a distribution defined by the subspace $T_1(x)=\{u\in T_0(x): g(u,U(x))=g(u,\phi U(x))=0\}$ of the tangent space T_xM at x. Taking X in T_0 and $Y=Z$ in T_1 in (3.8), we have

$$g(AY,U)g(AX,Y)-g(AY,\phi U)g(\phi AX,Y)=0.$$

Accordingly we get

$$g(AY,U)AY+g(AY,\phi U)A\phi Y=0, \quad Y\in T_1, \tag{3.10}$$

where we have used that $g(AY,\xi)=0$ and $g(A\phi Y,\xi)=0$. Since $A\phi U=\phi AU$, $g(AU,\phi U)=0$ and hence the vector field AU can be decomposed into

$$AU=\beta\xi+\gamma U+\delta' U_1, \tag{3.11}$$

where U_1 is a unit vector field in T_1, and γ and δ' are both smooth functions on M_0. Let M_1 be the subset of M_0 consisting of points x at which $\delta'(x) \neq 0$. Suppose that M_1 is not empty. By the forms $AU = \beta\xi + \gamma U + \delta' U_1$ and $A\phi U = \gamma\phi U + \delta'\phi U_1$, we have by (3.10)

$$g(Y, U_1)AY + g(Y, \phi U_1)A\phi Y = 0, \quad Y \in T_1$$

on M_1, which means that it satisfies $AU_1 = 0$ and hence $\delta' = 0$ by (3.11), a contradiction. Thus we get

$$\delta' = 0.$$

According to (3.9) and the above equation, we have

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi + \gamma U, \\ A\phi U = \gamma\phi U. \end{cases}$$

Hence we get by (3.8)

$$\begin{aligned} (h - \alpha - \gamma)\{&g(Y, U)g(AX, Z) + g(Z, U)g(AX, Y) \\ &- g(Y, \phi U)g(\phi AX, Z) - g(Z, \phi U)g(\phi AX, Y)\} \\ &+2d\beta(X)\{g(Y, U)g(Z, \phi U) + g(Z, U)g(Y, \phi U)\} \\ &-\beta\{g(Y, U)g(\nabla_X U, \phi Z) + g(Z, U)g(\nabla_X U, \phi Y) \\ &- g(Y, \phi U)g(\nabla_X U, Z) - g(Z, \phi U)g(\nabla_X U, Y)\} \\ = 0 \end{aligned} \tag{3.12}$$

for any vector fields X, Y and Z in T_0. Putting $Y = Z = U$ in this equation, we have

$$\beta g(\nabla_X U, \phi U) = \gamma(h - \alpha - \gamma)g(X, U), \quad X \in T_0.$$

Again, putting $Y = U$ and $Z = \phi U$ in (3.12), we get

$$d\beta(X) = -\gamma(h - \alpha - \gamma)g(X, \phi U), \quad X \in T_0. \tag{3.13}$$

In the case where $Y = U$ and Z in T_0 in (3.12), the above two equations give us

$$\beta g(\nabla_X U, \phi Z) = (h - \alpha - \gamma)\{g(AX, Z) - \gamma g(X, \phi U) g(Z, \phi U)\}$$

for any vector fields X and Z in T_0. Hence we obtain

$$\beta \nabla_X U = \beta\gamma g(X, \phi U)\xi + (h - \alpha - \gamma)\{\phi AX + \gamma g(X, \phi U)U\} \tag{3.14}$$

for any vector field X in T_0. Substituting (3.13) and (3.14) into (3.7), we get

$$g(\nabla_X A(Y), \xi) = (h - \gamma) g(\phi AX, Y) - g(A\phi AX, Y), \quad X, Y \in T_0.$$

Accordingly we obtain by (2.3)

$$g(\nabla_\xi A(X), Y) = (h-\gamma) g(\phi AX, Y) - g(A\phi AX, Y) + \frac{c}{4} g(\phi X, Y) \tag{3.15}$$

for any vector fields X and Y in T_0. Putting $X = U$ and $Y = \phi U$ in this equation, we get

$$\beta^2 - \gamma(h - 2\gamma) - \frac{c}{4} = 0,$$

where we have used that $g(\nabla_\xi A(U), \phi U) = \beta^2$. Again, putting $X = \phi U$ and $Y = U$ in (3.15), we have

$$\beta^2 + \gamma(h - 2\gamma) + \frac{c}{4} = 0.$$

Combining the above two equations, we get $\beta = 0$, a contradiction. Accordingly ξ is principal and hence $A\phi - \phi A = 0$ by (1.2). This completes the proof by Theorems A and B.

Remark. If M is a real hypersurface of type A in $M_n(c), c \neq 0, n \geqq 2$, then it satisfies $A\phi - \phi A = 0$ and S is η-parallel.(See [4] and [8].) Accordingly the converse of Theorem holds.

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