JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
27: 61 ~ 76, 1994

TIDAL EVOLUTION OF GLOBULAR CLUSTERS: THE EFFECTS OF GALACTIC TIDAL FIELD, DIFFUSION AND BLACK HOLES*

OH, KAP SOO
Department of Astronomy and Space Science, Chungnam National University
(Received Apr. 6, 1994; Accepted Apr. 18, 1994)

ABSTRACT

We investigate the dynamical evolution of globular clusters under the diffusion, the Galactic tide, and the presence of halo black holes. We compare the results with our previous work which considers the diffusion processes and the Galactic tide. We find the followings: (1) The black holes contribute the expansion of the outer part of the cluster. (2) There is no evidence for dependence on the orbital phase of the cluster as in our previous work. (3) The models of linear and Gaussian velocity distribution for the halo black holes do not show any significant differences in all cases. (4) The perturbation of black holes reduces the number of stars in lower energy regions. (5) There is a significant number of stars with retrograde orbits beyond the cutoff radius especially in the case of diffusion and the perturbation of black holes.

Key Words : stellar dynamics, globular clusters, black holes

I. INTRODUCTION

Globular clusters are ideal entities for systematic studies of stellar dynamics, since stars within a given cluster are coeval and have ages comparable to many Galactic orbital periods. On these timescales, clusters with 10^{5-6} stars undergo significant internal evolution provided that they are sufficiently compact. For the less centrally concentrated clusters, the Galactic tide affects both the structure and evolution of the outer regions. It has been suggested that both internal processes and the tidal interaction with the Galaxy have determined not only the dynamical evolution of globular clusters (Ostriker, Spitzer, & Chevalier 1972; Spitzer & Chevalier 1973) but also their survival probability (Fall & Rees 1977; Aguilar, Hut, & Ostriker 1988; Ostriker, Binney, & Saha 1989). For these theoretical reasons, the analysis of the tidal evolution of globular clusters is very important. There have been several previous attempts to investigate the tidal evolution of globular clusters (Keenan & Innanen 1975; Spitzer & Shull 1975; Aguilar & White 1985; Seitzer 1985).

Recently Oh, Lin, & Aarseth (1992, hereafter Paper I) presented a numerical scheme that is specifically constructed for investigating the tidal evolution of globular clusters. This scheme utilizes a Fokker-Planck approach as well as direct numerical integration of restricted three-body problem. In the inner regions of the cluster, stellar orbits are mapped with the cluster's gravitational potential and orbit averaged diffusion coefficients. In the outer regions, the Galactic tidal field is explicitly included in the direct integration. This scheme is not a general purpose tool and should be used with caution. But this scheme is ideally suited for studying tidal interaction between the Galaxy and closely bound globular clusters in which two-body relaxation is moderately efficient. Oh & Lin (1992, hereafter Paper II) presented the results which are obtained with this

* This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1992.

numerical method for various model parameters. We find following results. In clusters with negligible diffusion, clusters are tidally truncated to the theoretical tidal radius at perigalacticon. There is no apparent orbital phase dependence of the tidal radius for clusters with eccentric orbits. In clusters with moderately efficient two-body relaxation timescales, diffusion processes significantly modify the structure of the outer regions such that the cutoff radius may be comparable to the tidal radius at apogalacticon. Galactic tidal torque induces isotropy in the velocity dispersion of the outer regions of the cluster. For relaxed clusters, the velocity dispersion may be isotropic in the core, anisotropic in the envelope, and isotropic near the cutoff radius. Disk shocking is also very efficient for isotropizing the orbits of stars in the outer cluster regions. Stars with direct orbits are less stable, so that prolonged tidal interaction can lead to apparent retrograde rotation in the outer regions of the cluster. For clusters with highly anisotropic velocity dispersion, the tidally induced retrograde rotation effect may be extended into relatively small radii.

Lacey & Ostriker (1985) consider the idea that galaxy halos are composed of massive black holes as a possible resolution of problems of the composition of dark halos and the heating of stellar disks. They suggest the black holes with masses $M_H \sim 10^6 M_\odot$ to account for the amount of disk heating that is observed in our Galaxy. Black holes will perturb the orbits of stars in the globular cluster when black holes pass nearby globular clusters. If globular clusters experience such a close encounter with black holes, the evolution of globular clusters will be affected. We investigate the effect of these encounters with black holes.

The orbital kinematics, the equations of motion, the distribution function, the relaxation effect and the effect of black hole are described in §2. Model parameters for several different cases are given in §3. In §4 we present results of several cases. Finally we summarize the results in §5.

II. ORBITAL KINEMATICS

1. *Equations of Motion*

The size of globular clusters is typically two orders of magnitude smaller than their distance from the Galactic center. The stellar velocity dispersion of the clusters is also about two orders of magnitude smaller than the characteristic orbital velocity around the Galaxy. If the evolution of the cluster stars is computed in an inertial frame with respect to the Galactic center, the accumulated numerical truncation errors would significantly jeopardize the accuracy of the results. We find it advantageous to carry out the numerical computation in a comoving frame such that the guiding center of the frame is centered on the cluster core. Thus the frame corotates with the instantaneous orbital frequency where the azimuthal phase is defined with respect to the direction of the Galactic center. Consequently, this rotation rate of the comoving frame is not constant for the eccentric orbit.

We use the equations of motion for individual stars and the guiding center in the Appendix of Paper I for logarithmic Galactic potential in our simulations.

2. Distribution Function

In order to determine the cluster potential and evaluate diffusion coefficients, we need to specify a phase space distribution function for the stars. King (1966) adopted a simple distribution function for isotropic velocity dispersion

$$f(E) = C\,(e^{-\beta_e E} - 1), \qquad (1)$$

where C is a normalization constant, β_e a dimensionless model parameter, and E is the local orbital energy of the stars. From this distribution function, we can obtain static equilibrium solutions of the Boltzmann and Poisson equations which describe well the light distribution in globular clusters, where E vanishes at a finite outer radius. Hereafter this will be referred to as the 'cutoff radius' to distinguish it from the tidal radius which is determined from the tidal force.

3. Relaxation Effects

1) Diffusion Coefficients

Two-body encounters do not introduce significant changes in the stellar orbital angular momentum because the symmetry in the distribution about the trajectories leads to cancellation of the perturbations in transverse directions. This symmetry is preserved on all but the largest scales in the clusters.

In principle, the expectation value of energy perturbations over a time interval Δt is computed by integrating the appropriate distribution function over the energy and angular momentum phase space volume (Spitzer & Hart 1971a, b). However, a complete treatment would require evaluation of triple integrals which would be very time consuming. Following an approach by Marchant and Shapiro (1979), the two-body relaxation occurs mostly in the high-density cluster core where the relaxation timescale is short compared with the lifetime so that the velocity distribution is essentially relaxed and nearly isotropic. Such an approximation is adequate for computing the dynamical evolution prior to core collapse.

Using this approximation, the expectation value of energy perturbation, for a particle with energy E and angular momentum J, in one orbital period is given by equation (9) of Paper I.

2) Velocity Perturbations

Following the prescription by Spitzer and Thuan (1972), we introduce perturbations Δv_r and Δv_t to the radial and transverse velocities such that

$$\Delta v_r = X \left\{ < (\Delta v_{\parallel})^2 > \Delta t \right\}^{1/2} \frac{v_r}{v} = X \frac{\epsilon^2 v_r}{v^2}, \tag{2}$$

$$\Delta v_t = X \left\{ < (\Delta v_{\parallel})^2 > \Delta t \right\}^{1/2} \frac{v_t}{v} = X \frac{\epsilon^2 v_t}{v^2}, \tag{3}$$

where X denotes random numbers from a Gaussian distribution with zero mean and unit variance, $\Delta v_{\parallel}$ is the collisionally induced change of v in the direction parallel to the original trajectory, and ϵ_2 is the root mean square energy change induced by two-body interactions during one orbital period.

3) Perturbation due to Black Hole

Encounters with black holes will perturb the orbits of stars in the globular cluster. The dynamical friction will slow down the motion of black holes and consequently stars in the cluster will gain the kinetic energy. Then the cluster will expand and there will be more escapers. To see these predictions we consider several cases of velocity distribution of black holes.

III. MODEL

In order to increase the speed of numerical computation, we find it convenient to approximate the cluster potential in terms of a simple function of radius and the diffusion coefficients as a function of energy and angular momentum. We adopt the units $G=1$, central density $\rho_o=1$ and core radius $r_c=1$. This gives the central velocity dispersion of the cluster as $v_o=2\sqrt{\pi}/3$.

1. Internal Kinematic Models of the Cluster

In all our computations, we use an isotropic model which is generated from the King distribution function with a central potential $W_o=7.7$. The outer cutoff radius where the distribution function [eq. (1)] vanishes is about 55 r_c for this isotropic case. This is the case of globular cluster NGC2808 (Illingworth & Illingworth 1976). The half mass radii in this isotropic case are 5.8 r_c. When these results are appropriately scaled, the characteristic dynamical timescale, τ_c, is 1.0×10^6 $(R_c^3/M_5)^{1/2}$yrs, where R_c is the core radius in units of pc and M_5 is the clust mass in units of $10^5 M_\odot$. The structure of many globular clusters can be approximated by these models (Peterson & King 1975; Illingworth & Illingworth 1976; Webbink 1985).

We choose the critical radius to be 10 r_c. Interior to these radii, the Galactic tidal force contributes less than 1.7% of the gravitational force. At these radii, the diffusion coefficients retain more than 99% of the value for the case when the critical radius is set to infinity. The mass inside the critical radius is 68.3% of total mass. We generate an initial distribution of 5000 stars outside 10 r_c. The latter value is adopted to improve the statistics. When the stellar distribution is extrapolated to the cluster core, 5000 stars in the outer regions correspond to a total number of 1.57×10^4 stars.

We use several simple function to approximate various relevant quantities such as cluster's potential, diffusion coefficient and scattering angle inside the critical radii as in Paper I and Paper II.

2. Model Parameters

In order to determine the dependence of our results on various parameters of black hole, we computed five models for an isotropic velocity dispersion of cluster with Galactic tide, relaxation and influence of black holes.

We consider only spherically symmetric Galactic potentials in which the clusters' energy and angular momentum are conserved along their Galactic orbit. In all the cases below, we performed computations for Galactic potentials with a logarithmic spherically symmetric potential.

The strength of the Galactic tidal effect also depends on cluster's orbital eccentricity on which there is little observational information available. Statistical analyses of globular cluster kinematics (Frenk & White 1980) suggest large spread in clusters' orbital eccentricity. In this study, we examine an eccentric orbit with eccentricity 0.5.

We use the black holes with masses $M_H=10^6 M_\odot$ following Lacey & Ostriker (1985). We assume that the distribution of black holes is uniform near the globular cluster and the velocity dispersion is isotropic. We use two kinds of velocity distribution for the black holes, i.e. linear or Gaussian distribution.

We choose Model I5 and Model I6 from Paper II to investigate the effect of black holes on the evolution of the globular cluster. Both models use the isotropic velocity dispersions. We only consider the Galactic tidal effect in Model I5 but Model I6 is influenced by both the Galactic tide and internal diffusion processes. We tested the effect of softening parameter of the black hole in Model IB1LS and Model IB1L. The softening parameter in Model IB1LS is 0.01 and 0.5 in Model IB1L in unites of cluster core radii. All the other models use the value of 0.5. Model IB1LS, Model IB1L and Model IB1G are similar to Model I5, i.e. without diffusion processes. Model IB2L and Model IB2G consider diffusion processes like Model I6. Model IB1LS, Model IB1L and Model IB2L are for the case of linear velocity distribution of black holes and Model IB1G and Model IB2G for Gaussian velocity distribution. For computational convenience the black holes are generated randomly at the surface of the sphere with the radius of 1,100 core radii of the cluster which is 20 times of the initial cutoff radii of the cluster. At this distance the force ratio of black hole to the Galaxy on the cluster is less than 10^{-3}, so the effect of the initial appearance of black holes at this distance to the cluster member is negligible. The velocity distribution of the black holes will be similar to that of the globular cluster because they are halo objects. So we choose the mean velocity of the black holes is set to one-tenth of the velocity dispersion of the cluster because the origin of the coordinate is at the cluster center. We also assume the velocity dispersion of the black holes is similar to that of the cluster in the case of Gaussian velocity distribution with the same reason.

The parameters of five new models are summarized in Table 1 with two models studied in Paper II. The first column shows the model names. The total number of stars (N) is given in Column 2 in units of 10^5 where N determines the magnitude of the diffusion coefficients α. Column 3 lists the ratio of the mass of the Galaxy (M_G) to that of a cluster (M_c) in units of 10^6, where M_G refers to the Galactic mass inside the semimajor axis of the cluster. Constants for the diffusion coefficients (α) in equations (20) and (21) of Paper I are shown in Column 4. Column 5 gives the semimajor axis (a) of the orbit, in units of 10^4 core radii. The eccentricity (e), defined by equation (24) of Paper I, is given in Column 6. We define an orbital period, τ_o, as the time taken for one complete circular orbit or twice the time taken from apogalacticon to perigalacticon for an eccentric orbit. The crossing time, τ_c, is defined as the time required for a star moving with the rms velocity to traverse the half

-mass radius. The ratio of an orbital period to the crossing time is given in Column 7. The ratio of the half-mass relaxation time, τ_r, to the crossing time, defined by $N/\{26\log(0.4N)\}$ (Spitzer & Hart 1971a), where N is the total number of stars in the cluster, is given in Column 8. Column 9 contains the ratio of the relaxation time to an orbital period. Softening values for black holes is in Column 10 in units of the cluster core radii. Finally, column 11 summarizes the properties of each model.

Table 1. MODEL PARAMETERS.

Model (1)	N (10^5) (2)	M_G/M_c (10^6) (3)	$\sqrt{a}$ (10^{-2}) (4)	a (10^4) (5)	e (6)	τ_o/τ_c (7)	τ_r/τ_c (8)	τ_r/τ_o (9)	ϵ (10)	Comment (11)
I5	1.0	0.288	···	0.8731	0.5	174.8	···	···	···	Tide
IB1LS	1.0	0.288	···	0.8731	0.5	174.8	···	···	0.01	Tide, linear
IB1L	1.0	0.288	···	0.8731	0.5	174.8	···	···	0.5	Tide, linear
IB1G	1.0	0.288	···	0.8731	0.5	174.8	···	···	0.5	Tide, Gaussian
I6	1.0	0.288	4.1346	0.8731	0.5	174.8	835.7	4.781	···	Tide, diffusion
IB2L	1.0	0.288	4.1346	0.8731	0.5	174.8	835.7	4.781	0.5	Tide, diffusion, linear
IB2G	1.0	0.288	4.1346	0.8731	0.5	174.8	835.7	4.781	0.5	Tide, diffusion, Gaussian

The initial number of particles in all these models are 5000. I5 and I6 are listed for comparison. The ratio of velocity of the black hole to the velocity dispersion of the cluster is 0.1. And the velocity dispersion of the black holes is the same as that of the cluster.

IV. RESULTS AND DISCUSSION

In the simulations to be presented below, the evolution of each model is computed through to 30 orbital periods around the Galaxy. Typical models require about 50 cpu hours on a Sun IPC computer. We study a series of model parameters which allows an exploration of the parameter dependence.

Numerical simulations generate large quantities of data. The most useful information is contained in the spatial density ρ, surface density σ, the energy E and angular momentum J distributions, the ratio of transverse velocity v_t and radial velocity v_r, and the fraction of stars with retrograde orbits N_{ret} with respect to the orbital motion of the cluster. The correlation between energy and angular momentum indicates the eccentricity of stellar orbits in the cluster. In our analysis, the energy of each star is computed using the potential of an isolated cluster which is set to zero at infinity, while the potential of the Galaxy is neglected. In order to preserve statistical significance, we also set the spatial density and surface density at a given radial bin to zero if the number of stars in this bin is less than three.

1. Spatial Density

Spatial density plots for various models are shown in Figures 1, 2, and 3. Results near the final run are ploted to compare the effects of black holes. We also look into the dependence of the orbital distance. The panel (a) in each figure is at the time of perigalacticon passage after 29 orbits around the Galaxy. The panel (b) is at the time of 30th apogalacticon passage.

In Figure 1 we try to look at the dependence of softening values of black holes. Model IB1L and IB1LS use 0.5 r_c respectively. The density in the outer region is slightly higher in IB1LS than in IB1L in (a) but it is opposite in (b). So this is due to the statistical fluctuation. Hence the softening value of 0.01 and 0.5 does not make systematic differences. This is because the diffusion is dominated by many long distance encounters. So we use the softening value of 0.5 in the later models because the larger softening parameters can speed up the computation. The models of linear and Gaussian velocity distribution is shown in figure 2. The density of the outer region is slightly higher in IB1L and IB1G than in I5. So the black holes help the expansion of the cluster. But there is not a significant difference between linear and Gaussian velocity distribution. These trends are also shown in Figure 3, where internal diffusion is considered.

The number of stars between the theoretical tidal radius and 10 times of theoretical tidal radius in the case

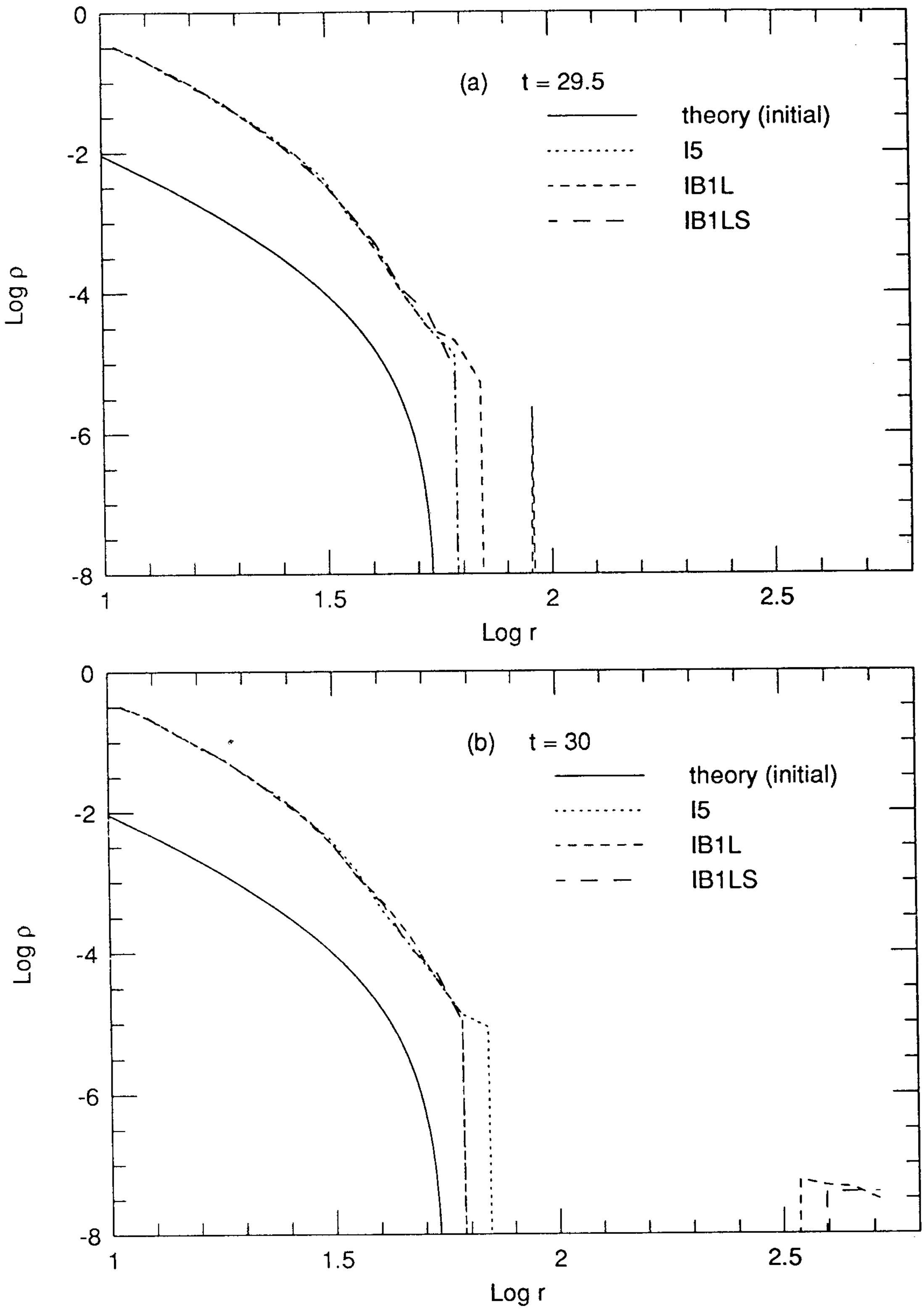


Fig. 1. Spatial density plot. Solid lines represent the theoretical profile of isotropic model with $W_o = 7.7$. Radii are in terms of core radius (r_c). Density is arbitrarily normalized. Model I5 is the case of the Galactic tide and for the comparison. Model IB1L has a softening value of 0.5 r_c for the black hole, and 0.01 r_c for Model IB1LS. Figure (a) is at the 29th passage of perigalacticon, and figure (b) at the 30th apogalacticon.

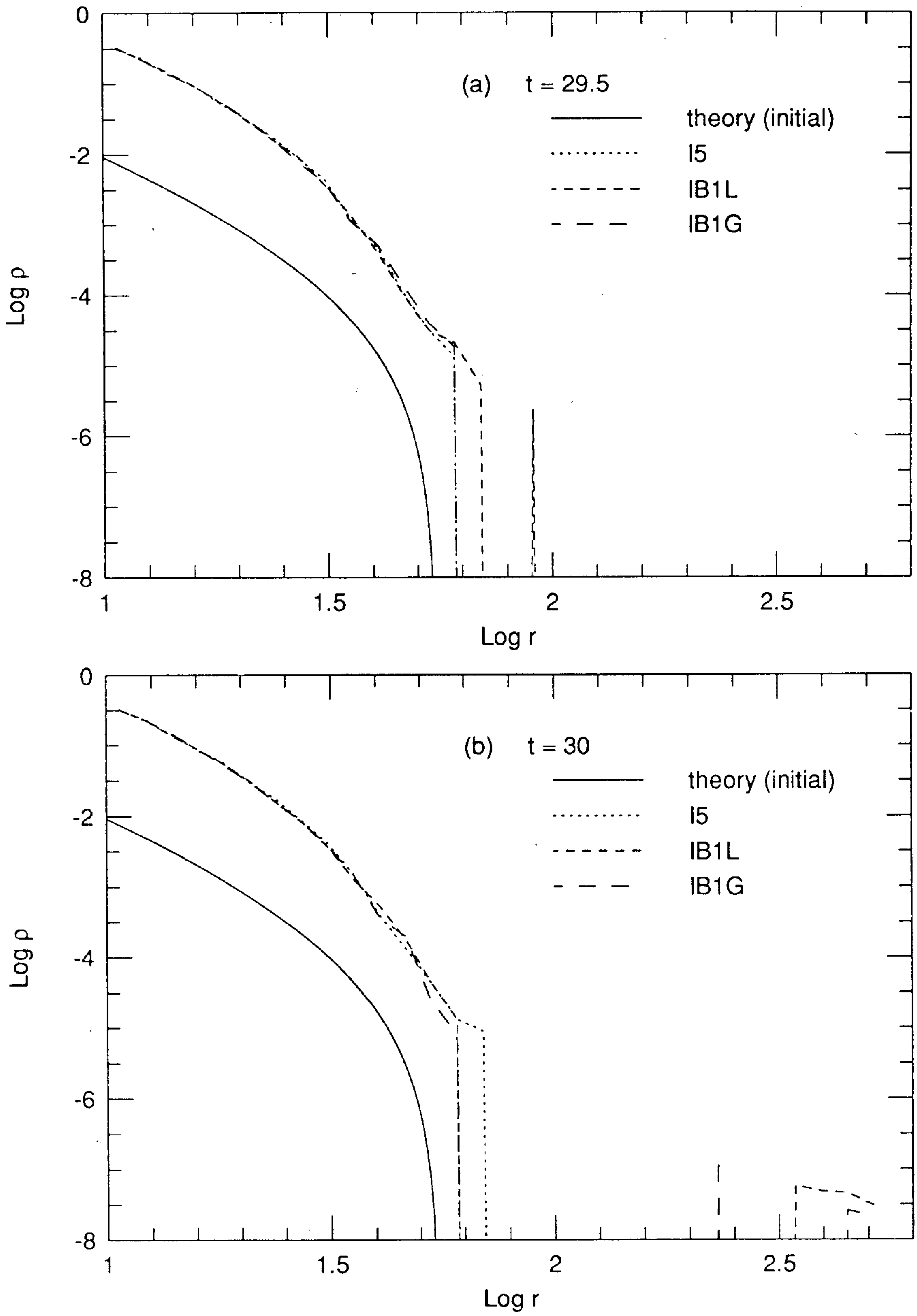
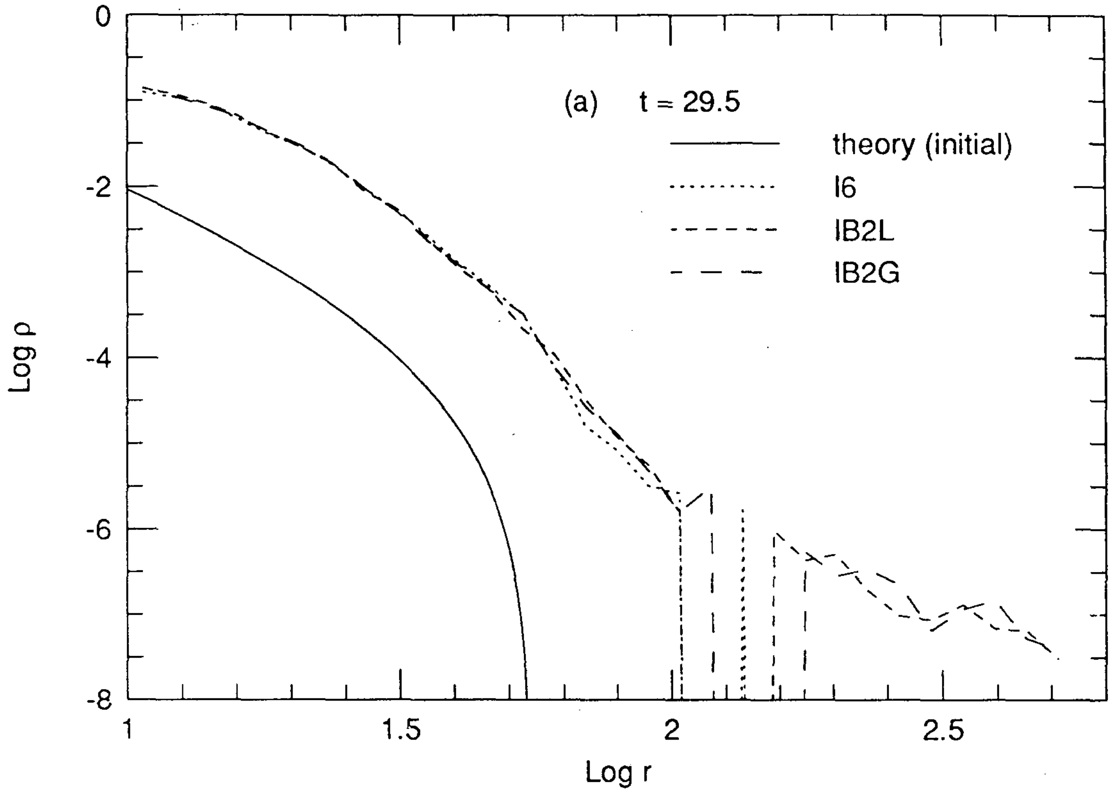


Fig. 2. Spatial density plot for Model IB1L and IB1G to compare the differences between linear and Gaussian velocity distributions of the black holes. Units are the same as in Fig. 1. The Galactic tide and perturbation of black holes are considered here.



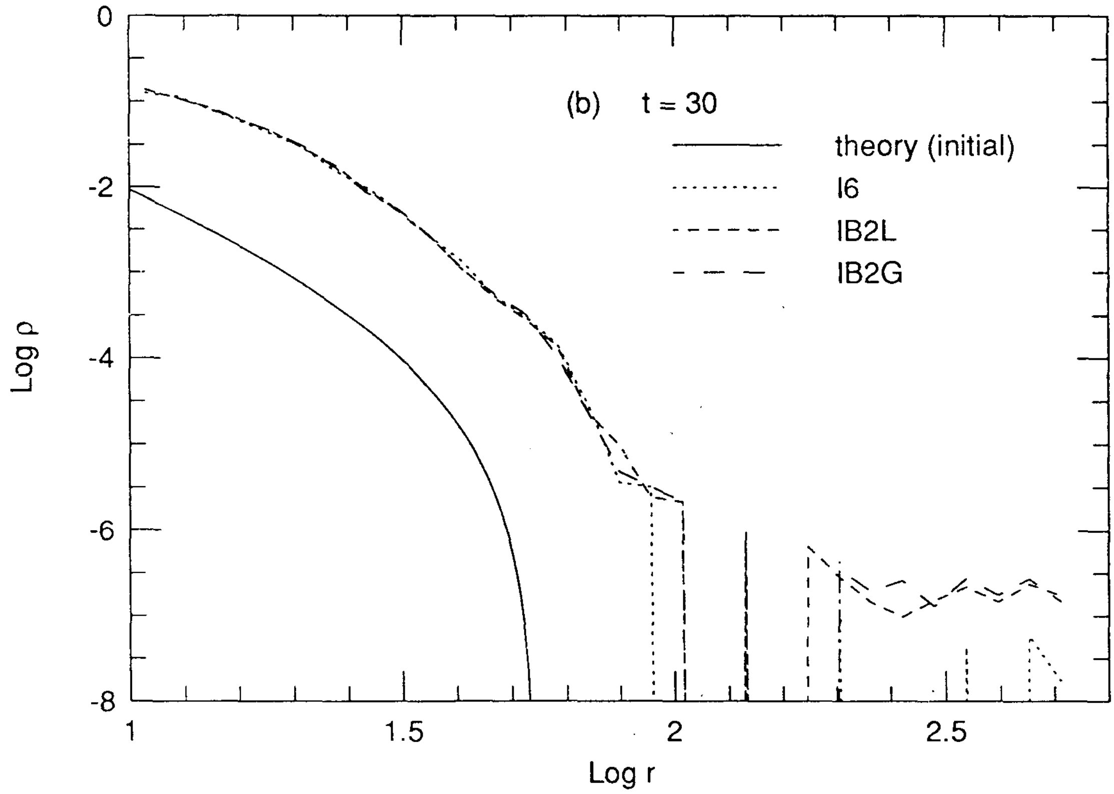


Fig. 3. Spatial density plot for Model IB2L and IB2G to compare the differences between linear and Gaussian velocity distributions of the black holes. Model I6 is the case of the Galactic tide and the diffusion for the comparison. Units are the same as in Fig. 1. The Galactic tide, the diffusion and perturbation of black holes are considered here.

of black hole perturbation is about 4 times larger than that in the case without black hole perturbation. So these will accelerate the disruption of globular clusters. However, it is hard to determine the accurate lifetime of globular clusters because the escape rate may not be the same with time.

2. Surface Density

Surface density plots for various models are shown in Figures 4, 5, and 6. Results near the final run are plotted in three different planes to compare the effects of black hole. We also look into the dependence of the orbital distance like in the case of spatial density. The (a) plot in each figure is at the time of perigalacticon passage after 29 orbits around the Galaxy. The (b) plot is at the time of 30th apogalacticon passage. These surface density plots show the similar trend to the spatial density plots. There is no dependence in the size of softening parameters and the black holes contribute the expansion of the cluster.

3. Energy and Angular Momentum

We also plot the individual energy and angular momentum in Figure 7 for various models. The angular momentum is normalized by the maximum angular momentum at a given enrgy. In this isotropic case, there is a broad spread of angular momentum for any given energy as expected. The evaporated stars migrate into the high-energy regions of phase space. This is because the main effect of two-body relaxation is a diffusion in energy. Stars diffuse to large radii as a result of energy gain rather than angular momentum gain. Consequently their orbits become more eccentric. Since two-body relaxation occurs near the center where the density has a maximum, orbits with large orbital semi-major axis would stop evolving unless they have a sufficiently large eccentricity to allow them to return to the cluster core. There are fewer stars in lower energy regions in the figure. This effect is prominant in the case of diffusion (b). This is because stars easily reach escape velocity with small perturbation due to black holes. There is also no significant difference between linear and Gaussian velocity distribution of black holes.

4. The ratio of Transverse and Radial Velocities

The behavior of velocity anisotropy is somewhat similar to that of the energy and angular momentum. Energy is related to the radius and angular momentum is also related to the ratio of the transverse and radial velocity. The diffusion process induces outward migration of stars with increasing orbital eccentricities. Thus, systems with relatively short relaxation timescales are more likely to have an anisotropic velocity dispersion at least in the intermediate regions of the cluster. At sufficiently large radii, the Galactic tide induces angular momentum transfer to the stars so that the transverse velocity becomes comparable to the radial velocity.

The ratio of transverse and radial velocities plots for various models are shown in Figure 8. The solid line indicates the position of the isotropic velocity dispersion of stars in the cluster. Stars above the solid line tend to have more circular orbit than those below the solid line. The ratio of stars above the line to stars below the line is higher in the case of black hole perturbation than without black hole perturbation.

5. The Fraction of Stars with Retrograde Orbits

Although the diffusion process is equally efficient for direct and retrograde orbits, the external tide differentially affects the stability of these orbit. The results in Paper II indicate that most of the stars exterior to the cutoff radius are mainly in prograde orbits except for the case with large diffusion coefficients. Near the theoretical tidal radius where the stellar orbital period in the cluster is comparable to that of the cluster around the Galaxy, the strong Galactic tidal perturbation during inferior conjunction is approximately in phase with the direct orbital motion and its prolonged action induces a considerable exchange of energy and angular momentum between the cluster's Galactic orbit and the stellar orbits in the cluster. Consequently, stars with direct orbits are most easily tidally removed from the cluster. In contrast, stars with retrograde motions have brief and frequent inferior conjunction such that the energy and angular momentum exchange rate due to the Galactic tidal torque is averaged to much smaller values. These orbits are much more stable. The fraction of stars with retrograde orbits for various models are shown in Figure 9. It shows that there is a significant number of stars with retrograde orbit beyond the cutoff radius especially in the case of diffusion (b). There are two reasons for

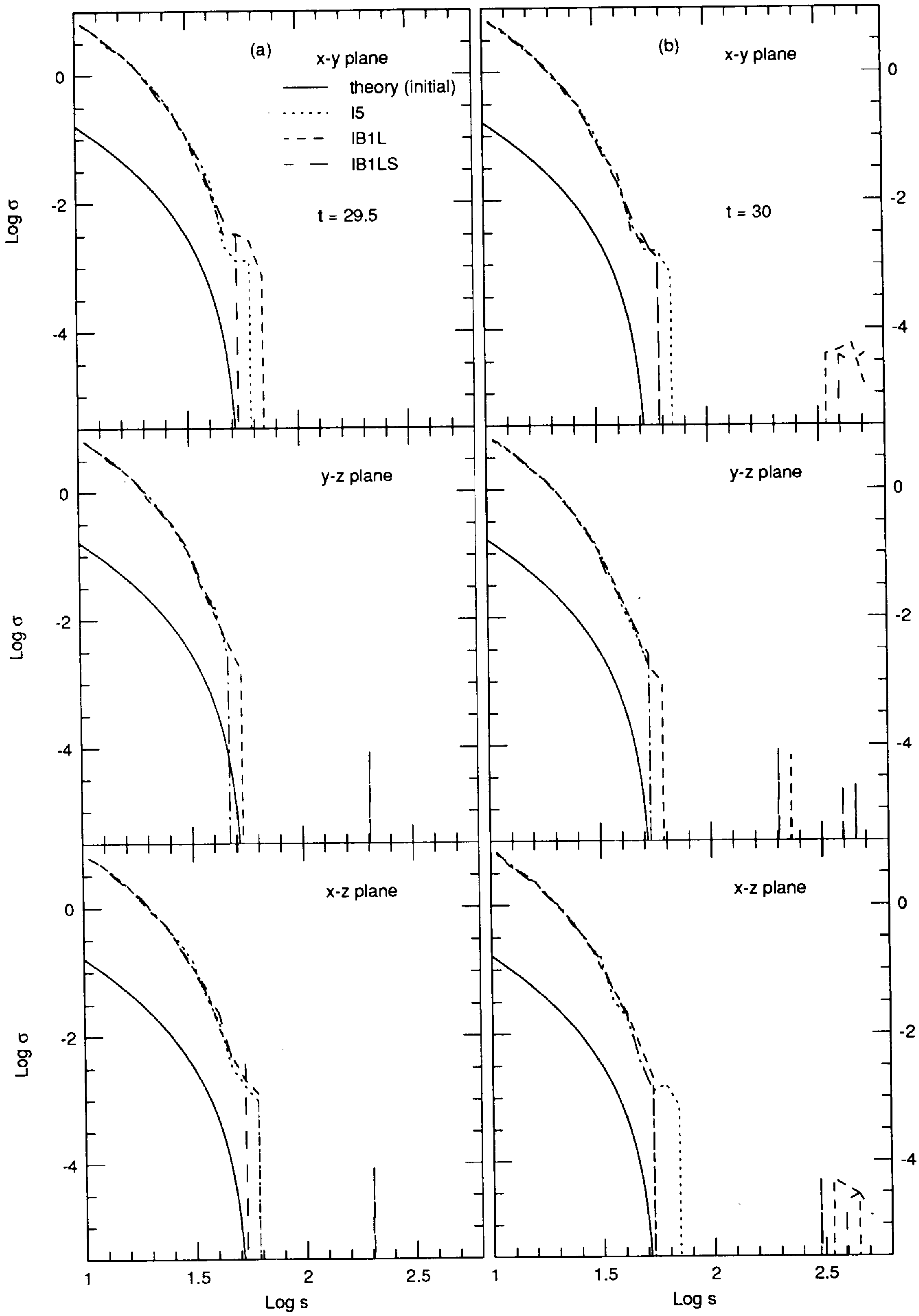


Fig. 4. Surface density plot on three different planes. Solid lines represent the theoretical profile of isotropic model with W_o = 7.7. Radii are in terms of core radius (r_c). Density is arbitrary normalized. Model IB1L has a softening value of 0.5 r_c for the black hole, and 0.01 r_c for Model IB1LS. Figure (a) is at the 29th passage of perigalacticon, and figure (b) at the 30th apogalacticon.

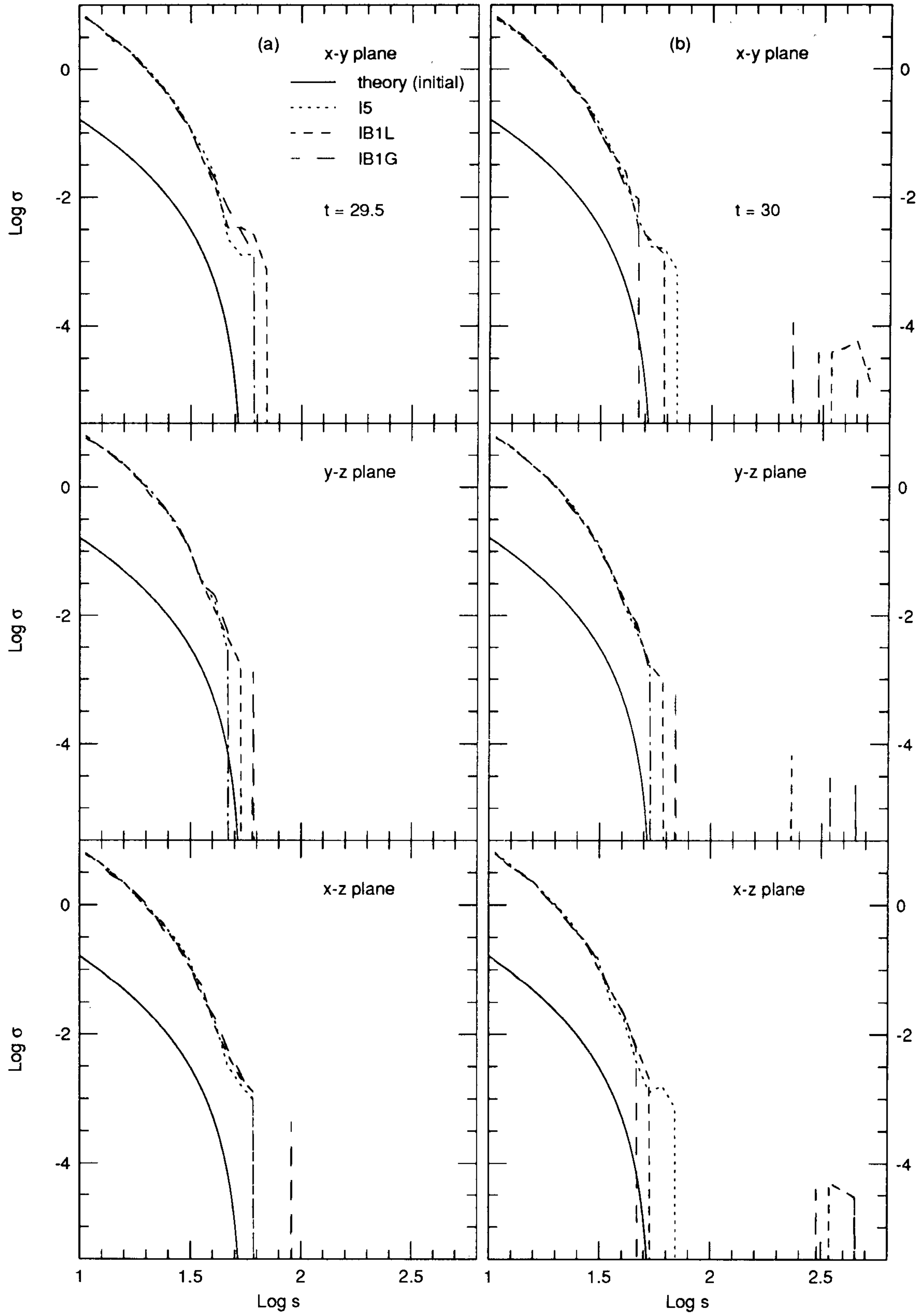


Fig. 5. Surface density plot for Model IB1L and IB1G to compare the differences between linear and Gaussian velocity distributions of the black holes. Units are the same as in Fig. 4. The Galactic tide and perturbation of black holes are considered here.

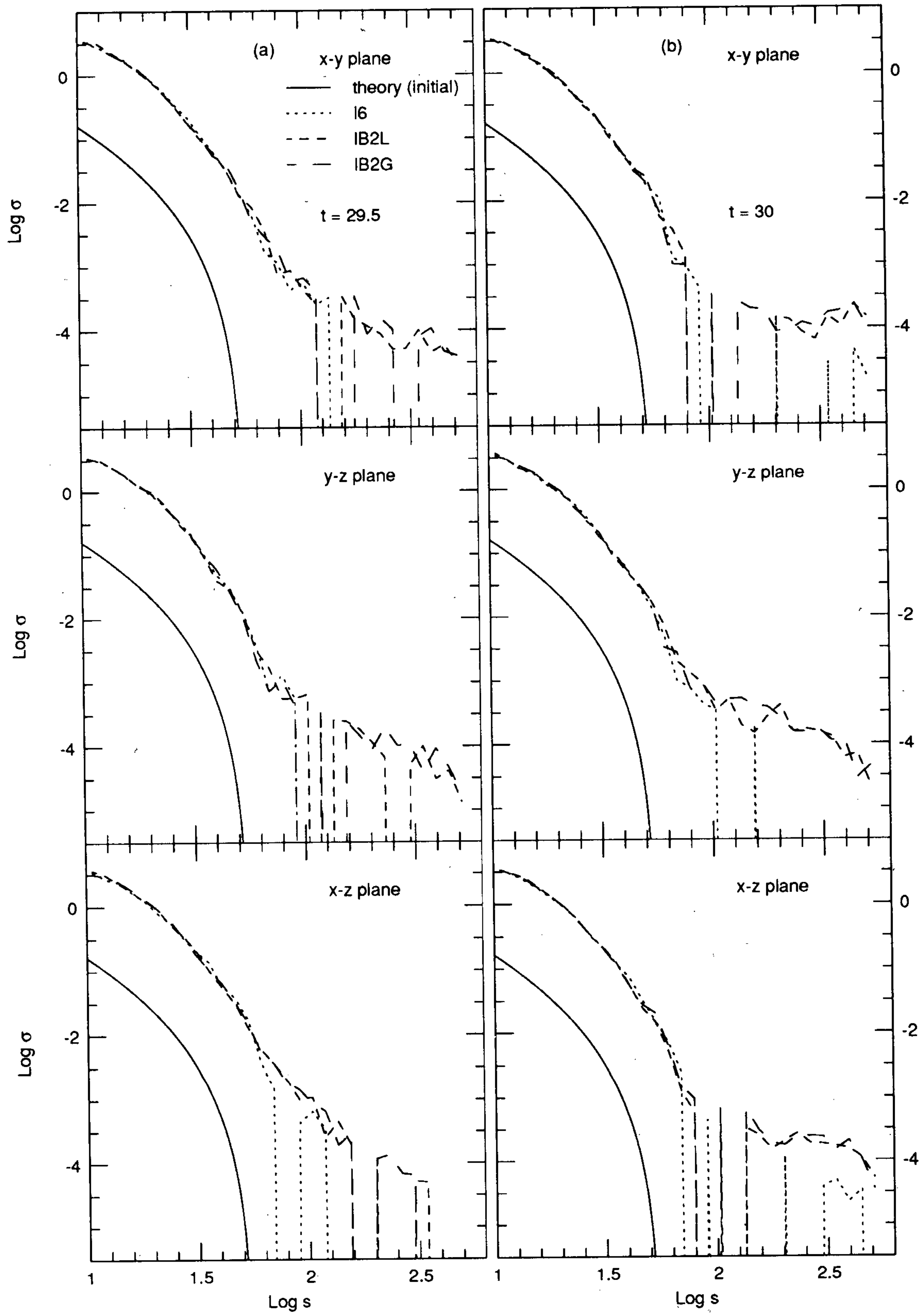


Fig. 6. Surface density plot for Model IB2L and IB2G to compare the differences between linear and Gaussian velocity distributions of the black holes. Units are the same as in Fig. 4. The Galactic tide, the diffusion and perturbation of black holes are considered here.

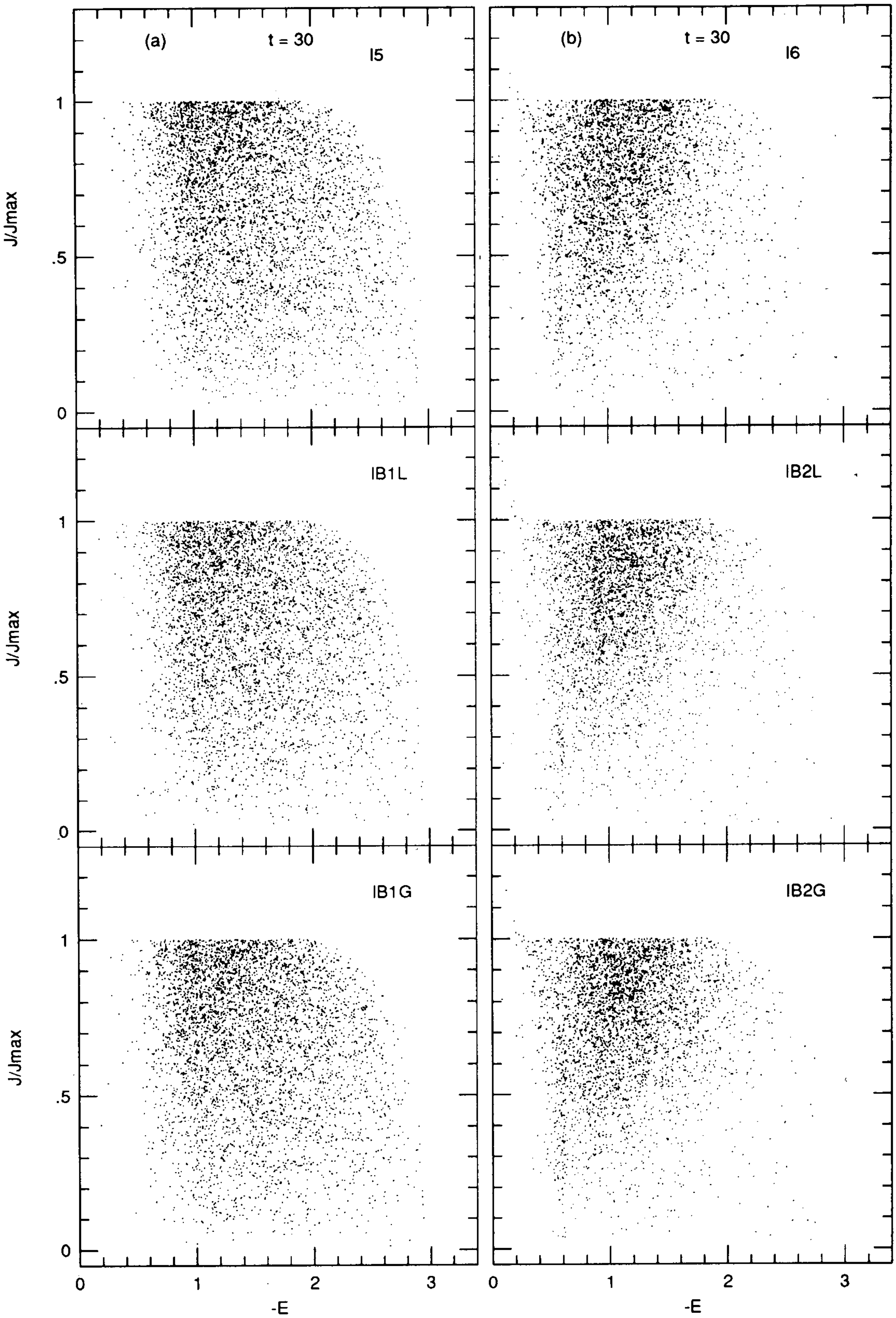


Fig. 7. Energy and angular momentum relation at the final stage of the simulation. The energy units are in terms of central velocity dispersion. The angular momentum is normalized by maximum angular momentum at a given energy.

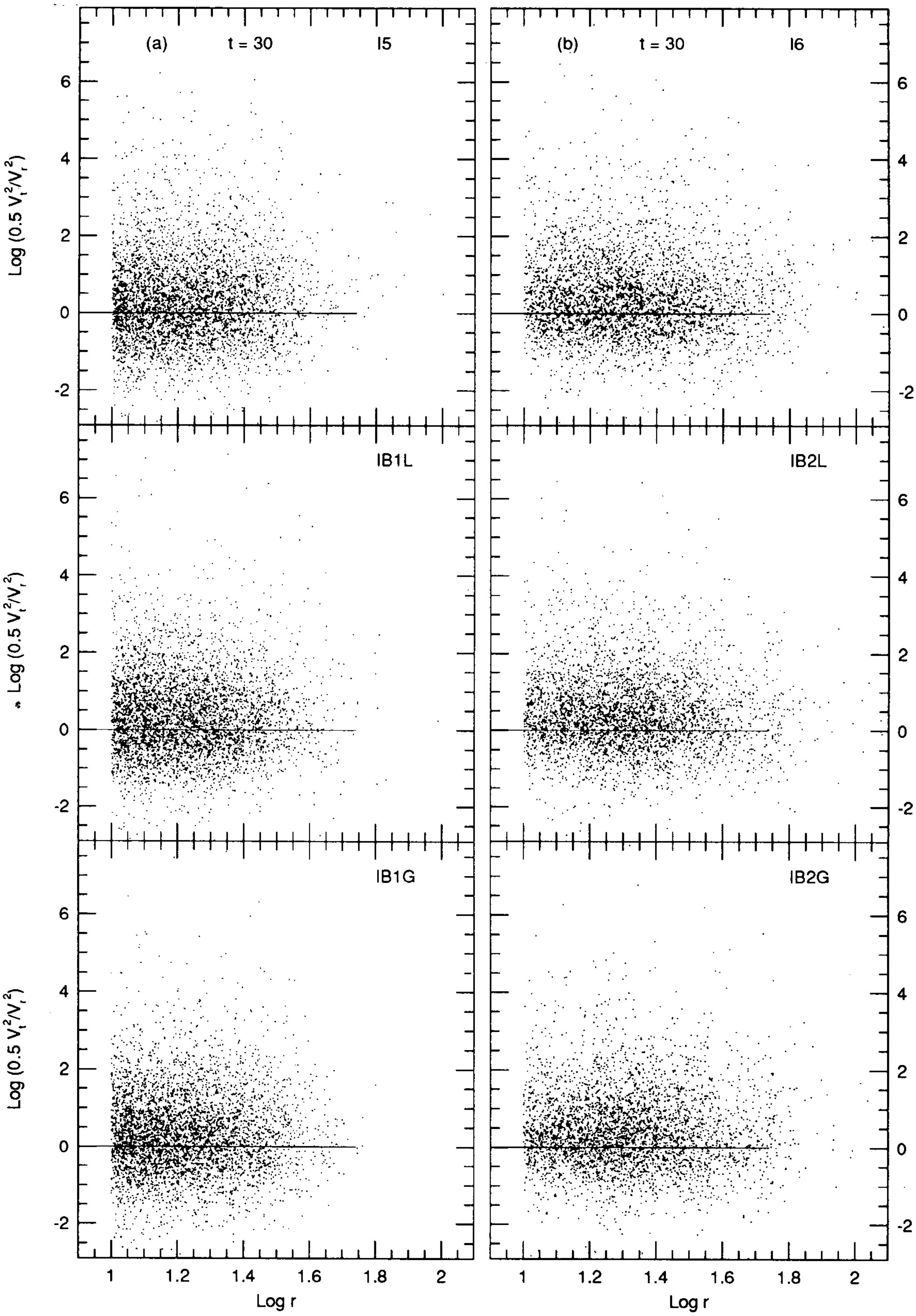


Fig. 8. Ratio of transverse and radial velocities at the final stage. In these velocity plots, the solid line represents the theoretical value of this ratio at various radii.

this result. First, the distribution of black holes is isotropic so that there is no distinction between prograde and retrograde orbits. Second, diffusion process does not have any preference to the direction of the orbit.

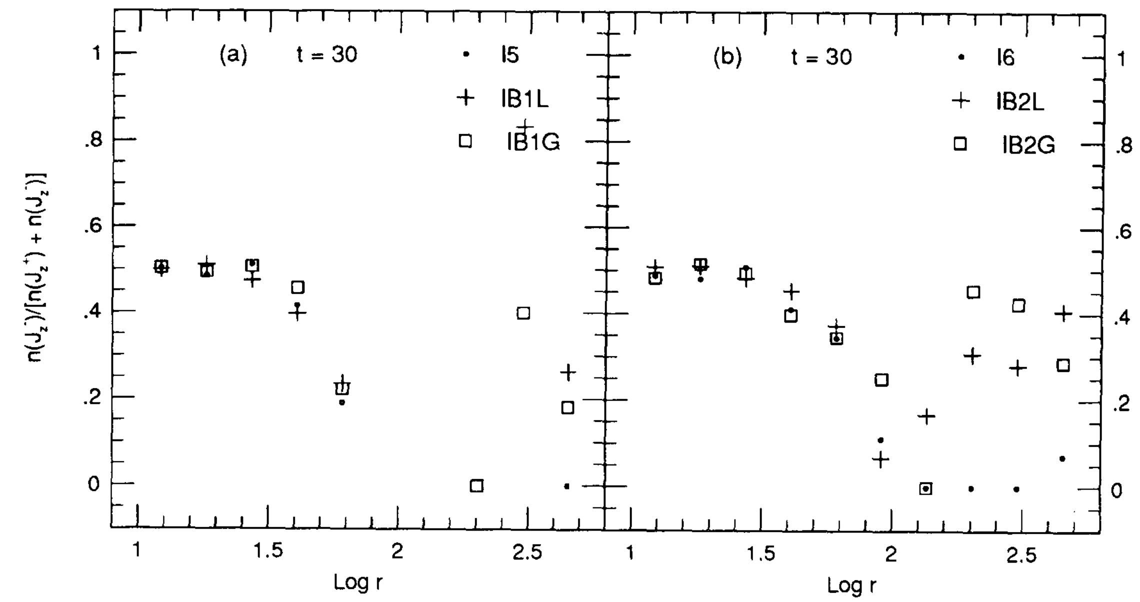


Fig. 9. Fraction of retrograde orbits.

V. SUMMARY

In this paper, we present numerical simulations of the tidal evolution of globular clusters under the influence of internal diffusion, the Galactic tide, and perturbation of halo black holes. This is the extension of our previous work (Paper I and Paper II) where internal diffusion and the Galactic tide are considered. So we focus on the effect of halo black holes in this paper.

First we examine the dependence of softening values of black holes. We find the softening value of 0.01 r_c and 0.5 R_c does not make any systematic differences in the spatial density and surface density. It is also shown that the black holes contribute the expansion of the outer part of the cluster. There is no evidence for dependence on the orbital phase of the cluster as in Paper I and Paper II. The spatial distribution of black holes is isotropic so the perturbation due to these black holes will reduce the orbital dependence. The models of linear and Gaussian velocity distribution for the halo black holes do not show any significant difference in all cases. There are fewer stars in lower energy regions in the case of diffusion and the perturbation of black holes. The ratio of stars of nearly circular orbit to stars of elongated orbit is higher in the case of black hole perturbation than without black hole perturbation. It shows that there is a significant number of stars with retrograde orbit beyond the cutoff radius especially in the case of diffusion and the perturbation of black holes. We conclude that halo black holes accelerate the expansion of the globular clusters and the escape rates. So the lifetime of the globular clusters will be reduced if black holes exist in the halo of the Galaxy.

REFERENCES

Aguilar, L. A., Hut, P. & Ostriker, J. P. 1988, ApJ, 335, 720

Aguilar, L. A. & White, S.D.M. 1985, ApJ, 295, 374
Fall, S. M. & Rees, M. J. 1977, MNRAS, 181, 37p
Frenk, C. S. & White, S.D.M. 1980, MNRAS, 193, 295
Illingworth, G. & Illingworth, W., 1976 ApJS, 30, 227
Keenan, D. W. & Innanen, K. A. 1975, AJ, 80, 290
King, I. R. 1966, AJ, 71, 64.
Lacey, C. G. & Ostriker, J. P. 1985, ApJ, 299, 633
Marchant, A. B. & Shapiro, S. L. 1979, ApJ, 234, 317
Oh, K. S., Lin, D.N.C. & Aarseth, S. J. 1992, ApJ, 386, 506
Oh, K. S. & Lin, D.N.C. 1992, ApJ, 386, 519
Ostriker, J. P., Binney, J. & P. Saha 1989, MNRAS, 241, 849
Ostriker, J. P., Spitzer, L. & Chevalier, R. A. 1972, ApJL, 176, L51
Peterson, C. J. & King, I. R. 1975, AJ, 80, 427
Seitzer, P. 1985, in IAU Symposium 113, Dynamics of Star Clusters, ed. J. Goodman & P. Hut (Dordrecht: Reidel), p. 343
Spitzer, L. & Chevalier, R. A. 1973, ApJ, 183, 565
Spitzer, L. & Hart, M. H. 1971a, ApJ, 164, 399
____________________. 1971b, ApJ, 166, 483
Spitzer, L. & Shull, J. M. 1975, ApJ, 201, 773
Spitzer, L. & Thuan, T. X. 1972, ApJ, 175, 31
Webbink, R. F. 1985, in IAU Symposium 113, Dynamics of Star Clusters, ed. J. Goodman & P. Hut (Dordrecht: Reidel), p. 541