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Analysis of Feedback Queues with Priorities

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Abstract

We consider single server feedback queueing systems with priorities. In the model, there are J stations and job classes. Jobs of class i arrive at station i according to a Poisson process, and have a general service time distribution. We derive the generating functions of the number of jobs at each station just after a busy period and the formula for the mean sojourn time that a specific tagged job spends at station j from its arrival to departure from the system.

1. Introduction

Feedback queueing models are useful for analyzing telecommunication systems, computer network systems and manufacturing systems. We consider single server feedback queueing systems with priorities. Jobs of class i arrive at station i from outside the system according to a Poisson process and are served according to the first come first served(FCFS) discipline. The service times of jobs at every station are arbitrarily distributed. After a service completion, the job at station i either departs from the system, or feeds back to the system and changes the station into k(i, k=1, ⋯, J). In the analysis of the statistics for these systems(e.g., mean sojourn times), it is important to know the workload of the system at the feedback arrival points. In this paper, first, we derive the generating functions of the number of jobs at each station just after a busy period conditioned on the arbitrary initial system state and investigate the statistics with regard to the system states. Secondly, we obtain the explicit formula for the mean sojourn time of a specific tagged job, which arrives at station i when the current state just prior to the arrival is n, spent at station j until the job departs from the system, (1≤ i, j≤J).

Disney [2] has been concerned with sojourn times in M/G/1 queues with instantaneous Bernoulli feedback. Van den Berg, et.

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al. [1] considered the system that each job requires N services. Fed back jobs return instantaneously, joining the end of the queue. They derived the set of linear equations for the mean sojourn times per visit which can be explicitly solved. Fontana and Diaz Berzosa[5] [6] derived the queue-length and waiting time distribution of an M/G/1 priority queueing model with more general feedback. Simon [11] considered the system with c types of jobs and m levels of priority. Type j jobs may require service N(j) times. He obtained the set of linear equations for the mean waiting times. Doshi and Kaufman [3]studied the sojourn time of a tagged job which has just completed its m^{th} pass in an M/G/1 queue with Bernoulli feedback. They also considered the model with multiple job classes. Epema [4] investigated the general single server (M/G/1) time-sharing model with multiple queues and job classes, priorities and feedback. Jobs are served in passes, receiving a complete quantum of service on every pass, or their remaining service demand, whichever is the less. If a job completes its service demand during the pass, it leaves the system. He derived a set of linear equations in the mean waiting times of the job passes for all classes and queues. Tcha and Pliska [12] studied the optimal scheduling problems of the multiclass queue with feedback.

The rest of this paper is organized as follows. In section 2, we describe the system in detail, and introduce notation and assumptions. In section 3, we investigate the statistics with regard to the system states. We derive the generating functions of the number of jobs at each station just after a busy period with the arbitrary initial system state and the mean values for these distributions. In section 4, we solve the mean sojourn time of a specific tagged job. The results from section 3 are used to obtain this statistic. Finally, section 5 contains the conclusion.

2. The model description

In this section we describe the structure of the queueing model which we treat in this paper(see figure 1). The model consists of a single server and J stations with infinite waiting rooms. Jobs of class i arrive at station i from outside the system according to a Poisson process $\{A_i(t : t \geq 0\}$ with rate $\lambda_i (i = 1, \cdots, J)$. The job at the station i is called the class i job, or simply the job i. The service time S_i of the class i jobs is independent and identically distributed with an arbitrary distribution function $F_i(t)$ and the Laplace Stieltjes Transform(LST)

$$\Psi_i(\Theta) = E\,[e^{-\Theta S_i}],\ i = 1, \cdots, J \text{ and } \Theta > 0. \qquad (2.1)$$

We assume the following overall disciplines :

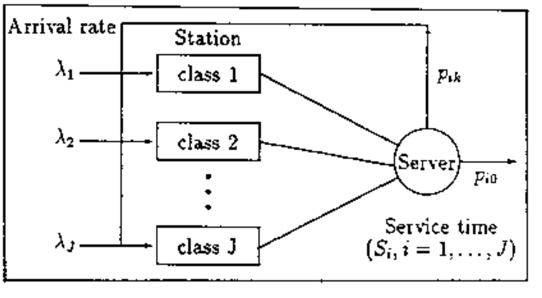

Figure 1. Feedback queue with J priorities

○ The jobs are preferentially served in ascending order of their classes. That is, the job j has priority over the job i if j is less than i.

○ The service discipline is preemptive over the class. When the job i arrives at the station i, all the jobs belonging to the classes between i+1and J are preempted from service.

The service discipline for each class is the FCFS discipline. That is, the station i serves jobs according to the first come first served basis if no jobs are in the stations from 1 to i−1. A preempted job resumes its service from the point of interruption. After receiving a service, the job i either departs from the system with probability $p_{i0}(=1-\sum_{k=1}^{J}p_{ik})$, or feeds back to the system and changes the class into k with probability p_{ik}(i, k=1, ⋯, J). Let $P_m=\{p_{ik} : i, k=1, \cdots, m\}$ (m=1, ⋯, J) be the feedback probability matrix. The arrival processes, the service times and the feedback processes are assumed to be independent of each other. Let T_{ij} be the total amount of service time that a job arriving at station i receives until the job departs from the system or leaves for one of the stations between j+1 and J for the first time (i, j=1, ⋯, J). Then,

$$T_{ij}=\begin{cases} S_i+T_{kj}, & \text{with probability } p_{ik},\ k=1, \cdots, j, \\ S_i, & \text{with probability } p_{ik},\ k=j+1, \cdots, J, \text{ or } 0 \end{cases} \tag{2.2}$$

The expected value of T_{ij} is

$$E[T_{ij}]=E[S_i]+\sum_{k=1}^{j}p_{ik}E[T_{kj}], \quad i, j=1, \cdots, J \tag{2.3}$$

We can obtain its solution in the vector form if$(I-P_j)^{-1}$ exists. Further we define the traffic intensity ρ_j in the following manner :

$$\rho_0=0,$$

$$\rho_j=\sum_{i=1}^{j}\lambda_i E[T_{ij}], \quad j=1, \cdots, J \tag{2.4}$$

The statistics are the server utilization of the jobs between class 1 and j. Then we make following assumptions.

Assumption 1

1. $P^n_J \to 0$ as $n\to\infty$
2. $\rho_J<1$.

The number of jobs in station i is denoted by n_i and its vector is denoted by $n=(n_1, \cdots, n_J)$. We call n the system state or simply the state. The interesting statistics in this paper are closely related to busy periods. Finally, we define the following statistics. Let

$$B^j(S)=\left\{\begin{array}{l}\text{the interval that begins with the} \\ \text{length of service S and ends,} \\ \text{for the first time after that,} \\ \text{when S has been performed} \\ \text{and the system is cleared of the} \\ \text{jobs between class 1 and} \\ \text{j, } j=1, \cdots, J\end{array}\right. \tag{2.5}$$

For notational convenience, let $B^0(S)=S$. In the usual queueing parlance, $B^j(S)$ is the exceptional first service busy period (EFSBP) composed of jobs between class 1 and j[13]. We will call $B^j(S)$ simply the class j busy period with the exceptional first service S. Its expected value is obtained by the usual method [13] :

$$E[B^j(S)]=E[S]/(1-\rho_j). \tag{2.6}$$

Further, we define

$$B^j(n)=\left\{\begin{array}{l}\text{the interval that begins with the} \\ \text{initial state n and ends,} \\ \text{for the first time after that,} \\ \text{when the system is cleared} \\ \text{of the jobs between class 1 and} \\ \text{j, } j=1, \cdots, J\end{array}\right. \tag{2.7}$$

Note that the jobs i(i=j+1, ⋯, J) are not performed in this period. We will call it simply the class j busy period with the initial state n. We now consider a set of jobs currently in the system and denote it by X. For example, if the m^{th} job i currently in the system belongs to the set, we express it as $(i, m_i) \in X$. The set of jobs that are in the system and are not in X when the system state is n is denoted by $\overline{X}$. We then define

$$B^j(n\,;X) = \begin{cases} \text{the interval that begins with the initial state n} \\ \text{and ends, for the first time after that, when the jobs belonging to X have been performed and the system is cleared of the jobs between class 1 and j except the jobs in } \overline{X}, \\ j=1, \cdots, J \end{cases} \quad (2.8)$$

The main difference between $B^j(n)$ and $B^j(x\,;X)$ is that the jobs (i, m_i) $(i=j+1, \cdots, J$ and $m_i=1, \cdots, n_i)$ can be performed in the latter period by appropriately choosing the set X. We will call it the class j buy period with the initial state (n ; X). Note that these EFSBPs are invariant for all work conserving service disciplines [10]. If we set $X_j=\{(i, m_i) : i=1, \cdots, j \text{ and } m_i=1, \cdots, n_i\}$, then $B^j(n)=B^j(n\,;X_j)$ for $j=1, \cdots, J$.

3. The system states after a class j busy period

In this section, we derive the generating functions of the states just after a class j busy period and the expected value of the states. These statistics are closely related to the busy periods. So we consider the states just after EFSBPs.

3.1 The generating functions of the system state

We derive the generating functions of the number of jobs at each station just after $B^j(n)$ [8]. Let $N_l^j(n)$ $(1 \le j < l \le J)$ be the number of jobs at the station l just after $B^j(n)$. This statistic consists of the jobs initially in station l, the jobs which arrived at station l from outside of the system and the jobs which arrived at station l by feedbacks. $B^j(n)$ is the sum of sub-busy periods generated by all jobs between station 1 and $j(j=1, \cdots, J)$. Hence, we have

$$B^j(n)=\sum_{i=1}^{j}\sum_{m=1}^{n_i} B^j(e_i\,;(i, m)) \qquad (3.1)$$

where e_i is the i^{th} unit vector. Then we have

$$N_l^j(n)=n_l+\sum_{i=1}^{j}\sum_{m=1}^{n_i} N_l^j(e_i\,;(i, m)),$$
$$1 \le j < l \le J \qquad (3.2)$$

All random variables on the right-hand side of the equation are mutually independent. Now we define :

$$Q_i^j(z)=E\left[\prod_{l=j+1}^{J} z_l^{N_l^j(e_i)}\right]$$

where $z=(z_1, \cdots, z_J)$.

Then the generating function of the number of jobs at each station just after $B^j(n)$ is as follows.

$$Q^j(z \mid n) \equiv E\left[\prod_{l=j+1}^{J} z_l^{N_l^j(n)}\right]$$
$$=\left(\prod_{l=j+1}^{J} z_l^{n_l}\right)\left[\prod_{i=1}^{j} (Q_i^j(z))^{n_i}\right] \qquad (3.3)$$

Similarly as in the derivation of the ordinary busy period LST in the M/G/1 queue, $Q_i^j(z)$ can be obtained by considering a service discipline which serves nonpreemptively the jobs according to the last come first served basis. Since the busy period $B^j(n)$ is invariant for all work conserving service disciplines, $N_l^j(n)$ is also invariant for such disciplines. $N_l^j(e_i)$ can be decomposed into the number of jobs arriving at the station l in the sub-busy periods initiated by the jobs which arrived at each station during T_{ij}. Thus we have

$$N_l^j(e_i) = 1_l(T_{ij}) + A_l(T_{ij}) + \sum_{k=1}^{j}\sum_{m=1}^{A_k(T_{ij})} N_l^j(e_k;(k,m)),$$
$$1 \le i \le j < l \le J \qquad (3.4)$$

where

$$1_l(T_{ij}) = \begin{cases} 1, & \text{if the job i is in the station l just after } T_{ij} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}(T_{ij}) = (1_1(T_{ij}), \cdots, 1_J(T_{ij}))$. Then we have

$$Q_i^j(z \mid T_{ij}, \mathbf{1}(T_{ij})) \equiv E\left[\prod_{l=j+1}^{J} z_l^{N_l^j(e_i)} \mid T_{ij}, \mathbf{1}(T_{ij})\right]$$
$$= \left(\prod_{l=j+1}^{J} z_l^{1_l(T_{ij})}\right) E\left[\prod_{l=j+1}^{J} z_l^{A_l(T_{ij})} \mid T_{ij}\right]$$
$$\prod_{k=1}^{j} E\left[\prod_{m=1}^{A_k(T_{ij})}\prod_{l=j+1}^{J} z_l^{N_l^j(e_k;(k,m))} \mid T_{ij}\right] \qquad (3.5)$$

Since the arrival processes are Poisson, it can be shown that,

$$Q_i^j(z \mid T_{ij}, \mathbf{1}(T_{ij})) = \left(\prod_{l=j+1}^{J} z_l^{1_l(T_{ij})}\right) e^{-T_{ij}\eta_j(z)} \qquad (3.6)$$

where $\eta_j(z) = \sum_{l=j+1}^{J}\lambda_l(1-z_l) + \sum_{k=1}^{j}\lambda_k(1-Q_k^j(z))$.

Let X(t) denote the station where the tagged job(served at first) is at time t. Let T_j be the time until the tagged job departs from the system or enters one of the stations between j+1 and J for the first time. Further we define

$$M_i^j(z,\eta) \equiv E\left[\left(\prod_{l=j+1}^{J} z_l^{1_l(T_j)}\right) e^{-\eta T_j} \mid X(0) = i\right]$$

Then we obtain M_i^j by conditioning on $X(S_i)$:

$$M_i^j(z,\eta) = \Psi_i(\eta)\left\{\sum_{k=1}^{j} p_{ik} M_k^j(z,\eta) + \sum_{k=j+1}^{J} p_{ik} z_k + p_{i0}\right\},$$
$$(1 \le i \le j \le J). \qquad (3.7)$$

where $\Psi_i(\eta)$ is LST of S_i. Then we have

$$Q_i^j(z) = M_i^j(z, \eta_j(z)) \qquad (3.8)$$
$$= \Psi_i(\eta_j(z))\left\{\sum_{k=1}^{j} p_{ik} M_k^j(z, \eta_j(z)) + \sum_{k=j+1}^{J} p_{ik} z_k + p_{i0}\right\} \qquad (3.9)$$

where $1 \le i, j \le J$. Thus, $Q^j(z \mid n)$ obtained from (3.3).

3.2 The expected value of the system state

We derive the expected value of the system states just after a class j busy period with the initial state n. Let $n' = (n_1', \cdots, n_J')$ be the system state just after a class j busy period with the initial state n. Then the expected value of $n_l'(E[n_l' \mid n])$ can be obtained from the generating function $Q^j(z \mid n)$ as follows :

$$E[n_l' \mid n] = \frac{\partial Q^j(z \mid n)}{\partial z_l} \Big|_{z=1} \qquad (3.10)$$

where $1 = (\overbrace{1, 1, \cdots, 1}^{n})$. If we let

$$\frac{\partial Q_i^j(z)}{\partial z_l} \Big|_{z=1} = N_{il}^j, \ (1 \le i \le j)$$

where N_{il}^j is interpreted as the mean number of jobs at station l just after a class j busy period generated by a single job of class i. Then, from (3.3) and (3.9), we obtain

$$E[n_l' \mid n] = \begin{cases} 0, & 1 \le l \le j, \\ n_l + \sum_{i=1}^{j} n_i N_{il}^j, & j < l \le J \end{cases} \qquad (3.11)$$

where

$$N^j_{il}=\lambda_i E[S_i]+p_{il}+\sum_{k=1}^{j}\lambda_k E[S_i]+p_{ik})N^j_{kl},$$
$$(1\leq i\leq j< l\leq J) \qquad (3.12)$$

Thus, in matrix notation,N^j_{il} $(i=1, \cdots, j)$ can be obtained from the following equation.

$$N^j_l=(I-\overline{T}_j\Lambda_j)^{-1}(I-P_j)^{-1}a^j_l,$$
$$(1\leq j\leq i< l\leq J) \qquad (3.13)$$

where we define :

$$a^j_l=(\lambda_l E[S_1]+p_{1l}, \cdots, \lambda_l E[S_j]+p_{jl})^t,$$
$$(1\leq i\leq l)$$
$$N^j_l=(N^j_{1l}, N^j_{2l}, \cdots, N^j_{jl})^t, \ (i\leq l\leq J)$$
$$\overline{T}_j=\begin{pmatrix} E[T_{1j}] \\ \vdots \\ E[T_{jj}] \end{pmatrix}$$
$$\Lambda_j=(\lambda_1, \cdots, \lambda_j)$$

where the superscript t denotes the transpose. The existence of the inverse matrix $(I-P_j)^{-1}$ is supposed in Assumption 1. It can easily be shown that $(I-\overline{T}_j\Lambda_j)^{-1}$ exists by Theorem 7.3 in [7].

4. The mean sojourn time

In this section, we derive the mean sojourn time $W_{ij}(n)$ of the tagged job i, which arrives at station i when the initial state just prior to the arrival is n, spent at station j until the job departs from the system, $(1\leq i, j\leq J)$. We postulate $W_{0j}(n)=0$ for convenience. Note that we exclude the tagged job from the initial state. The mean sojourn time is composed of two parts. The first one is the mean time from the arrival at station i to completion of its first service at the station. We call this the initial mean sojourn time and denote by $W_i(n)$. The second one is the mean sojourn time after spending the initial mean sojourn. In this paper, we assume that no job is in the middle of service at the feedback arrival of the tagged job i and at the completion of the initial sojourn time. Thus we can express the mean sojourn as a function of the system state. Then, $W_{ij}(n)$ can be written as follows :

$$W_{ij}(n)=\begin{cases} E[W_{Kj}(n^*) \mid n], & i\neq j, \\ W_j(n)+E[W_{Kj}(n^*) \mid n], & i=j, \end{cases} \qquad (4.1)$$

where K is the station number for which the tagged job i leaves after staying at station i and n^* is the system state just after the initial sojourn time.

4.1 The initial sojourn time

The initial mean sojourn time $W_i(\cdot)$ of the tagged job i is composed of two parts [9]. The first one is delay D^1_i prior to entering service for the first time from the arrival and the second one is the service completion time D^2_i of the tagged job i being the time from when service begins on tagged job i until service is completed. Let $X_i(i=1, \cdots, J)$ be the set of jobs composed of jobs belonging to the classes between 1 and i which are initially in the system. Then D^1_i is the first time until the jobs belonging to X_i have been performed and the system is cleared of jobs between classd 1 and $i-1$. Thus D^1_i is the class $i-1$ busy period with the initial state $(n;X_i)$. D^2_i begins just after D^1_i , and ends when the service time S_i has been completed and the system is cleared of jobs between class 1 and $i-1$. Thus D^2_i is also the class $i-1$ busy period with the exceptional first service S_i, Hence, from (2.5) and (2.8), we have

$$W_i(n)=E[B^{i-1}(n\ ;\ X_i)+B^{i-1}(S_i)],$$

$$i=1,\ \cdots,\ J \tag{4.2}$$

The EFSBP is the sum of the sub-EFSBPs starting with every job in X_i. Notice that the order of service within X_i is irrelevant. These expected value are easily obtained from (2.6). Then we have

$$W_i(n)=\sum_{l=1}^{i} n_l \frac{E[T_{l,i-1}]}{(1-\rho_{i-1})}+\frac{E[S_i]}{(1-\rho_{i-1})},$$
$$i=1,\ \cdots,\ J \tag{4.3}$$

where ρ_i is given in (2.4).

4.2 The system state after the initial sojourn time

In order to solve (4.1), we also need to know the state just after the initial sojourn time. In this subsection, we derive the generating functions of the state just after the initial sojourn time and obtain the expected value of the states. We now use the results from section 3 to obtain these statistics. Let $\bar{N}_l^{i-1}(e_i)$ $(1 \le i \le l \le J)$ be the number of jobs(except the tagged job) at station l just after $B^{i-1}(S_i)$. Then,

$$\bar{N}_l^{i-1}(ei)=A_l(S_i)+\sum_{k=1}^{i-1}\sum_{m=1}^{A_k(S_i)} N_l^{i-1}(e_k\,;(k,m)),\quad i\le l\le J \tag{4.4}$$

Similar to the previous result (3.9), its generating function can be obtained as follow :

$$\bar{Q}_i^{i-1}(z)\equiv E\Big[\prod_{l=i}^{J} z_l^{\bar{N}_l^{i-1}(e_i)}\Big] = \Psi_i(\eta_{i-1}(z)) \tag{4.5}$$

where $\eta_{i-1}(z)=\sum_{l=i}^{J}\lambda_l(1-z_l)+\sum_{k=1}^{i-1}\lambda_k(1-Q_k^{i-1}(z))$. The empty sum, which often occurs at $i=1$, is defined to be 0 from now on. Let $N_l^{i-1}(n\,;X_i)$ be the number of jobs at the station l just after $B^{i-1}(n\,;X_i)$. We can also obtain its generating function as follows :

$$Q^{i-1}(z\mid n\,;X_i)\equiv E\Big[\prod_{l=i}^{J} z_l^{N_l^{i-1}(n\,;x_i)}\Big] =\Big(\prod_{l=i}^{J} z_l^{n_l}\Big)\prod_{(i,m_i)\in X_i} Q_i^{i-1}(z) \tag{4.6}$$

where $Q_i^{i-1}(z)$ can be obtained from (3.9). Then from (4.2), (4.5) and (4.6), the generating function of the state n^* is

$$\Psi_i(z\mid n)\equiv E\Big[\prod_{l=i+1}^{J} z_l^{n_l^*}\mid n\Big] = Q^{i-1}(z\mid n, X_i)\bar{Q}_i^{i-1}(z) =\Big(\prod_{l=i+1}^{J} z_l^{n_l}\Big)\Big[\prod_{k=1}^{i}(Q_k^{i-1}(z))^{n_k}\Big]\Big[\bar{Q}_i^{i-1}(z)\Big] \tag{4.7}$$

The expected value of n_l^* $(E[n_l^*\mid n])$ can be obtained from (4.7) as follows :

$$E[n_l^*\mid n]=\frac{\partial\Psi_i(z\mid n)}{\partial z_l}\Big|_{z=1} \tag{4.8}$$

where $\mathbf{1}=(\overbrace{1,1,\ \cdots,\ 1}^{n})$. If we let

$$\frac{\partial\bar{Q}_i^{i-1}(z)}{\partial z_l}\Big|_{z=1}\equiv\bar{N}_{il}^{i-1}$$

where $1\le j\le i\le l\le J$. Then we have

$$E[n_l^*\mid n]=\begin{cases}0, & l=1,\ \cdots,\ i-1,\\ \sum_{j=1}^{i} n_j N_{jl}^{i-1}+\bar{N}_{il}^{i-1}, & l=i\\ n_l+\sum_{j=1}^{i} n_j N_{jl}^{i-1}+\bar{N}_{il}^{i-1}, & l=i+1,\ \cdots,\ J\end{cases} \tag{4.9}$$

where N_{jl}^{i-1} and $\bar{N}_{il}^{i-1}$ can be obtained from (3.13) and (4.5).

4.3 The formula of the mean sojourn time

As we have seen above, the initial sojourn times are the sum of the sub-EFSBPs starting with the tagged job i and every initial job in X_i. From (4.3), the initial sojourn time is a linear function of the initial system state. We define the following vector :

$$F^i \equiv (f^i_1, \cdots, f^i_J)^t \in R^{J\times 1},\ i=1,\cdots,J$$

where the superscript t denotes the transpose and

$$f^i_l = \begin{cases} \dfrac{E[T_{li-1}]}{(1-\rho_{i-1})} & l=1,\cdots,i \\ 0, & l=i+1,\cdots,J \end{cases}$$

$$g^i = \frac{E[S_i]}{(1-\rho_{i-1})}$$

Then we may generally assume the following :

$$W_i(n) = nF^i + g^i,\ i=1,\cdots,J \qquad (4.10)$$

Also, the state just after the initial sojourn time is the sum of the number of jobs starting with the tagged job i and every initial job in X_i. From (4.9), the expected number of jobs just after the initial sojourn time with the initial state n is linear in n. Thus, we can generally express it as

$$E[n^* \mid n] = nN^i + \overline{N}^i \qquad (4.11)$$

for some matrix $N^i \in R^{J\times J}$ and some vector $\overline{N}^i \in R^{1\times J}$. We make again the assumptions that are obtained above.

Assumption 2.

$$W_i(n) = nF^i + g^i, \qquad (4.12)$$

$$E[n^* \mid n] = nN^i + \overline{N}^i \qquad (4.13)$$

where $1 \le i \le J$. Of course, the assumption is satisfied by the FCFS discipline. Then we derive the mean sojourn times $W_{ij}(\cdot)$ under the above assumptions. Now we define the following matrices. Fix the index $J(1 \le j \le J)$.

$$F = (0, \cdots, 0, F^{j^t}, 0, \cdots, 0)^t \in R^{(J\cdot J)\times 1},$$

$$Q = \begin{bmatrix} p_{11}\tilde{I} & p_{12}\tilde{I} & \cdots & p_{1J}\tilde{I} \\ p_{21}\tilde{I} & p_{22}\tilde{I} & \cdots & p_{2J}\tilde{I} \\ \vdots & & & \\ p_{J1}\tilde{I} & p_{J2}\tilde{I} & \cdots & p_{JJ}\tilde{I} \end{bmatrix} \in R^{(J\cdot J)\times(J\cdot J)},$$

$$N = \begin{bmatrix} N^1 & & 0 \\ & N^2 & \\ 0 & & N^J \end{bmatrix} \in R^{(J\cdot J)\times(J\cdot J)},$$

where the superscript t denotes the transpose and $\tilde{I}$ is an identity matrix in $R^{J\times J}$. We suppose that $(I-NQ)^{-1}$ exits where I is an identity matrix in $R^{(J\cdot J)\times(J\cdot J)}$. Then we can define

$$H = \begin{pmatrix} H_{1j} \\ \vdots \\ H_{Jj} \end{pmatrix} = (I-NQ)^{-1} F \in R^{(2J\cdot J)\times 1}$$

Further we define

$$\overline{h} = \begin{bmatrix} \overline{N}^1 \sum_{l=1}^{J} p_{1l} H_{lj} \\ \vdots \\ \overline{N}^{j-1} \sum_{l=1}^{J} p_{j-1l} H_{lj} \\ g^j + \overline{N}^j \sum_{l=1}^{J} p_{jl} H_{lj} \\ \overline{N}^{j+1} \sum_{l=1}^{J} p_{j+1l} H_{lj} \\ \vdots \\ \overline{N}^J \sum_{l=1}^{J} p_{Jl} H_{lj} \end{bmatrix} \in R^{J\times 1}$$

From Assumption 1, $(I-P_J)^{-1}$ exists. Then we can define

$$h = \begin{pmatrix} h_{1j} \\ \vdots \\ h_{Jj} \end{pmatrix} = (I-P_J)^{-1} \overline{h} \in R^{J\times 1},$$

Now the following theorem is proved.

Theorem Let $1 \le j \le J$ and Assumptions 1 and 2 hold. Further if we assume that $(I-NQ)^{-1}$ exist, then

$$W_{ij}(n) = nH_{ij} + h_{ij},\ (1 \le i \le J)$$

is a solution of the equation (4.1).

(Proof) First, let

$$W_{ij}(n) = nH_{ij} + h_{ij},\ (1 \le i \le j \le J) \qquad (4.14)$$

Then, it is sufficient to show that (4.14) is a solution of (4.1). Substituting (4.12), (4.13) and (4.14) into (4.1),
For $i \neq j$,

$$\begin{aligned} &E[W_{Kj}(n^*) \mid n] \\ &= E[n^* H_{Kj} + h_{Kj} \mid n] \\ &= \sum_{l=1}^{j} p_{il}\{E[n^* \mid n] H_{lj} + h_{lj}\} \\ &= \{nN^i + \overline{N}^i\} \sum_{l=1}^{J} p_{il}H_{lj} + \sum_{l=1}^{J} p_{il}h_{lj} \\ &= n\left\{N^i \sum_{l=1}^{J} p_{il}H_{lj}\right\} + \left\{\overline{N}^i \sum_{l=1}^{J} p_{il}H_{lj} + \sum_{l=1}^{J} p_{il}h_{lj}\right\} \\ &= W_{ij}(n) \end{aligned}$$

The last equation follows from the definition of the constants H_{ij} and h_{ij}, that is,

$$H_{j} = N^i \sum_{l=1}^{J} p_{il}H_{lj} \quad (4.15)$$

$$h_{ij} = \overline{N}^i \sum_{l=1}^{J} p_{il}H_{lj} + \sum_{l=1}^{J} p_{il}h_{lj} \quad (4.16)$$

Hence $W_{ij}(n)$ satisfies the equation (4.1).
For $i = j$,

$$\begin{aligned} &W_j(n) + E[W_{Kj}(n^*) \mid n] \\ &= nF^j + g^j + E[n^* H_{Kj} + h_{Kj} \mid n] \\ &= nF^j + g^j + \sum_{l=1}^{J} p_{jl}\{E[n^* \mid n] H_{lj} + h_{lj}\} \\ &= nF^j + g^j + \{nN^j + \overline{N}^j\} \sum_{l=1}^{J} p_{jl}H_{lj} + \sum_{l=1}^{J} p_{jl}h_{lj} \\ &= n\left\{F^j + N^j \sum_{l=1}^{J} p_{jl}H_{lj}\right\} + \left\{g^j + \overline{N}^j \sum_{l=1}^{J} p_{jl}H_{lj} + \sum_{l=1}^{J} p_{jl}h_{lj}\right\} \\ &= W_{jj}(n) \end{aligned}$$

The last equation follows from the definition of the constants H_{jj} and h_{jj}, that is,

$$H_{jj} = F^j + N^j \sum_{l=1}^{J} p_{jl}H_{lj} \quad (4.17)$$

$$h_{jj} = g^j + \overline{N}^j \sum_{l=1}^{J} p_{jl}H_{lj} + \sum_{l=1}^{J} p_{jl}h_{lj} \quad \square \quad (4.18)$$

5. Conclusion

In this paper, we are concerned with feedback queueing systems with priorities. We derived the generating functions of the system states just after a busy period and investigated the statistics with regard to the system states.

Also, under Assumption 1 and Assumption 2 and the assumption regarding the existence of the inverse matrix, we obtained the formula for the mean sojourn time that a specific tagged job spends at station j from its arrival to departure.

The methlodogy given in this paper will be widely applicable to the analysis of the multiclass queueing system.

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