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A Planar Geodesic Constrained On the Maximum Curvature and with Prescribed Initial and Terminal Directions : An Optimal Control Approach

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Abstract

In this article, a planar geodesic (2-dimensional minimum length curve between two points) on which the maximum curvature is constrained and with prescribed initial and terminal directions is studied. A generic problem is formulated by the minimum-time optimal control problem in free terminal time. It is shown that the optimal path (G^2) may contain a singular arc or not and that the general types of G^2 can be classified into the 3 classes of control sequences. Finally, the explicit form of G^2 is derived geometrically as well as algebraically form the main theorem of this article.

1. Introduction

In this article, a planar geodesic which implies two-dimensional minimum length curve between initial and terminal points is considered. Particularly, our major concern is devoted to the curves which are constrained on their maximum curvature and the initial and the terminal directions of the curves are prescribed.

A seminal work related to this theme is found in by Dubins [2]. He proposes several conditions for the existance of R^2-geodesic and shows what types of it must be. But in showing the general types of R^2-geodesic, he uses long and winding passages of algebraic and geometric tools. He does not mention the explicit form of the solution in generic fashion. By applying the optimal control scheme, we develop an unified approach to this problem. And by deriving the explicit from of solution, the solution charactreistics of the generic problem can be inverstigated rigorously.

2. Formulations

Let $x(t) \in R^2$ be the position of a particle at time t. The $v(t) = \dot{x}(t)$ and $a(t) = \ddot{x}(t)$ denote the velocity and the

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acceleration vectors at time t, respectively. Let v be the constant speed of a particle (i. e. $\| v(t) \| = v, \ \forall t$) and assume that the control is bounded with its maximum curvature R (i.e. $\| a(t) \| \le R, \ \forall t$).

Suppose that a particle starts at an initial point x_0 and pursues a continuous path to a terminal point x_1. Further, suppose that its velocity vectors at the initial and the terminal points are prescribed with v_0 and v_1, respectively. We are interested in a minimum length curve with its prescribed initial and terminal directions(hereafter, we will denote the optimal path by G^2). (see [Fig. 1])

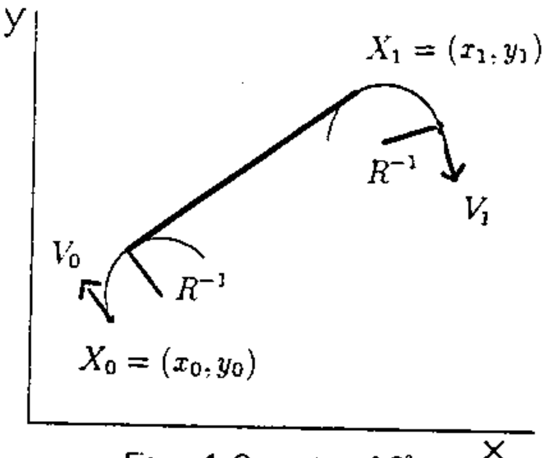


Figure 1. Geometry of G^2

To obtain G^2 with given parameters, an potimal control approach will be applied. The first version of optimal control formulation to obtain G^2 is the following [P1];

[P1]

$$Max_{a(t),\, T} \int_0^T -v dt \tag{1}$$

$$s.t. \quad \dot{x}(t) = v(t) \tag{2}$$

$$\dot{v}(t) = a(t) \tag{3}$$

$$\| v(t) \| = v, \ \forall t \tag{4}$$

$$\| a(t) \| \le R, \ \forall t \tag{5}$$

$$x(O) = x_0 \tag{6}$$

$$x(T) = x_1 \tag{7}$$

$$v(0) = v_0 \tag{8}$$

$$v(T) = v_1 \tag{9}$$

[P1] can be described as the '*minimum-time*' problem with our inital setting. In this case, the minimum time problem has an identical property with the minimum length problem because of the constancy of the norm of the velocity vector. Unfortunately, $a(t) \in R^2$, it is hard to handle the problem in its original form, since [P1] contains two -dimensional control function $a(t) \in R^2$. For this reason, we will translate [P1], into the equivalent formulation which contains one-dimensional control. Consequently, the two dimensional velocity vector ($v(t) \in R^2$) is replaced by the one dimecsional directed angle ($\theta(t) \in R^1$), (see [Fig. 2]).

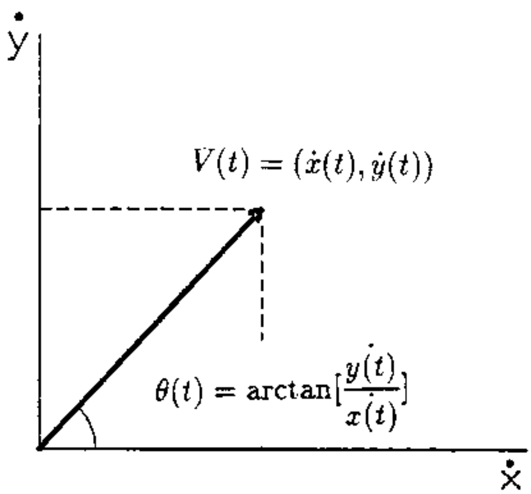


Figure 2. The concept of directed angle, $\theta(t)$

$$\theta(t) = arctan\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right) \tag{10}$$

According to the one-dimensional state variable, $\theta(t)$, the first and the second coordinate of the velocity vector can be written as,

$$v(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = v \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \quad (11)$$

and the control bound on the maximum curvature of the G^2 can be translated into the condition,

$$|\dot{\theta}(t)| \leq \frac{R}{v} \quad (12)$$

In (12), the constancy of the speed(v) allows the direct translation of the bound on '*maximum curvature*' into the bound on '*maximum steering angle*'. Without any loss of generality, we may assume both the magnitude of the speed(v) and the maximum curvature (R) to be unity. We can also transform the initial and the terminal positions to $x_0 = (0,\ 0)$, $x_1 = (l,\ 0)$ respectively. Formulation [P2] is the simplest version of the generic problem to solve G^2 (see [Fig. 3]). It can be classified into a minmum-time optimal control problem with control bound and fixed terminal values in free terminal time.

[P2]

$$Max_{u(t),\ T} \int_0^T -1dt \quad (13)$$

$$s.t.\ \dot{x}(t) = \cos\theta(t) \quad (14)$$

$$\dot{y}(t) = \sin\theta(t) \quad (15)$$

$$\dot{\theta}(t) = u(t) \quad (16)$$

$$|u(t)| \leq 1,\ \forall t \quad (17)$$

$$x(0) = (0,\ 0) \quad (18)$$

$$x(T) = (l,\ 0) \quad (19)$$

$$\theta(0) = \theta_0 \quad (20)$$

$$\theta(T) = \theta_1 \quad (21)$$

$$where\ 0 \leq \theta_0,\ \theta_1 \leq 2\pi \quad (22)$$

Remark 1 Since $v = 1$ the optimal solution T^* of [P2] is the length of the optimal path

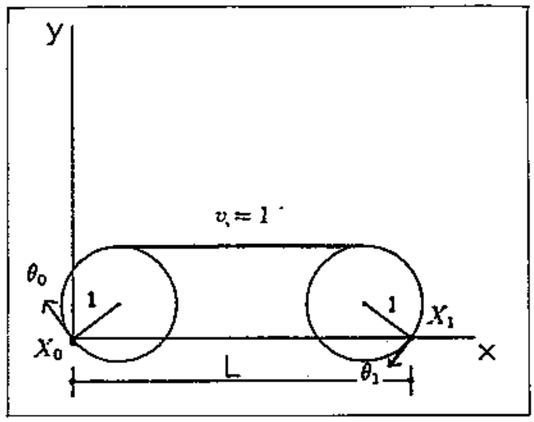


Figure 3 Geometry of the modified problem

$x^*(t) = [x^*(t),\ y^*(t)],\ 0 \leq t \leq T^*$

Remark 2 $|u(t)|\ (= |\dot{\theta}(t)|)$ equals to the curvature of x(t).

Its proof is trivial.

3. Optimality conditions for G^2

The necessary conditions for G^2 in [P2] can be derived from Pontryagin's Maximum principle ([1], [3]). the Hamiltonian is given by

$$H = -1 + \lambda_1 \cos\theta + \lambda_2 \sin\theta + \mu u \quad (23)$$

where the adjoint (co-state) variables, λ_1, λ_2 and μ, are time-varying functions. The Lagrangian for [P2] is defined by

$$L = H + \omega_1(u + 1) - \omega_2(u - 1) \quad (24)$$

To obtain a maximum of the objective functional, the Lagrangian is maximized at each point in time with respect to u and T. The necessary conditions are obtained as follows;

1. The feasibilty conditions : state equation (14), (15), (16). Initial and terminal conditions (18)-(21).

2. The adjoint equations :

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0 \quad (25)$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = 0 \quad (26)$$

$$\dot{\mu} = -\frac{\partial H}{\partial \theta} = \lambda_1 \sin\theta - \lambda_2 \cos\theta \quad (27)$$

3. Kuhn-Tucker conditions :

$$\frac{\partial L}{\partial u} = \mu + \omega_1 - \omega_2 = 0 \quad (28)$$

$$\omega_1(u+1) = 0 \quad (29)$$

$$\omega_2(u-1) = 0 \quad (30)$$

$$\omega_1,\ \omega_2 \geq 0 \quad (31)$$

4. The transversal conditions :

$$H(T) = 0 \quad (32)$$

Since [P2] is an autonomous system, its Hamiltonian must be constant over time. With this result, we can extend the transversality condition (32) to,

$$H(t) = -1 + \lambda_1 \cos\theta + \lambda_2 \sin\theta + \mu u \equiv 0,\ \forall t \quad (33)$$

From(23), the optimal control is *'bang-bang'*, since the Hamiltonian is linear in the control function u. i.e.

$$u^*(t) = \begin{cases} -1, & H_u = \mu(t) < 0 \\ ?, & H_u = \mu(t) = 0 \\ 1, & H_u = \mu(t) > 0 \end{cases} \quad (34)$$

In (34), $u^*(t)$ is determined by the sign of the adjoint variable $\mu(t)$. In case $\mu(t) = 0$. however, $u^*(t)$ is indeterminate. There are two possibilities with the occurrence of this singularity. If $\mu(t) = 0$ only for isolated instants. there is no problem. In such case, $u^*(t)$ becomes pure bang-bang which has its control elements-1 and 1 only. However, if $\mu(t) = 0$ over some period of time, the choice of $u^*(t)$ is not determined in the usual way. In the latter case, the value of $u^*(t)$ is said to be singular. With the occurrence of the singularity, the extremal path candidates for G^2 can be classified into two classes.

Def *Extremal path candidate for* G^2 is a path which satisfies the necessary conditions of the generic problem [P2]. The set of all extremal path candidates for G^2 is denoted by Γ^*.

Let $\Gamma^*_S \subset \Gamma^*$ be the set of the extremal path candidates for G^2 which contains singular arcs and let $\Gamma^*_n \subset \Gamma^*$ be the set of those without singular arcs. Obviously, $\Gamma^* = \Gamma^*_S \cup \Gamma^*_n$ and $\Gamma^*_S \cap \Gamma^*_n = \phi$. In obtaining the elements of Γ^*_n, the necessary conditions (1-4) are all we have. But for Γ^*_S, the additional necessary conditions must be added during the singularity period of time, $t \in [t_1, t_2]$

5. The singularity conditions :

$$H_u = \mu(t) = 0,\ t \in [t_1, t_2] \quad (35)$$

$$\dot{H}_u = \dot{\mu}(t) = 0,\ t \in [t_1, t_2] \quad (36)$$

$$\ddot{H}_u = \ddot{\mu}(t) = 0,\ t \in [t_1, t_2] \quad (37)$$

$$\frac{\partial}{\partial u}(\ddot{H}_u) \geq 0,\ t \in [t_1, t_2] \quad (38)$$

Equation (35)-(37) describe the singularity conditions and inequality (38) is called the Legendre-Clebsch condition.

Def *The set of trinomial control sequences.* denoted by Σ, is the set of sequences of which elements are in $\{-1, 0, 1\}$.

Let $\Sigma^* \subset \Sigma$ be the set of trinomial control sequences corresponding to Γ^*. And let $\Sigma^*_S \subset \Sigma^*$ and $\Sigma^*_n \subset \Sigma^*$ be the set of trinomial control sequences corresponding to Γ^*_S, Γ^*_n, respectively. Also, $\Sigma^* = \Sigma^*_S \cup \Sigma^*_n$. Denote $\sigma^*_S \in \Sigma^*_S$ and $\sigma^*_n \in \Sigma^*_n$ by the element of the set of trinomial control sequences.

Theorem 1 If $G^2 \in \Gamma^*_S$, then $\sigma^*_S \in \sum^*_S$ is is a subsequence of the following two classes,

Π-class : (1, 0, 1), (−1, 0, −1)
χ-class : (1, 0, −1), (−1, 0, 1)

Proof : It is sufficient to show that the singular control is zero (i.e. $u^*(t) = 0,\ t \in [t_1, t_2]$) and that there must be at most one nonsingular control both before and after the singularity.
From (33) and (35),

$$H = -1 + \lambda_1 \cos\theta + \lambda_2 \sin\theta \equiv 0.$$
$$\forall t \in [t_1, t_2]$$

Substituting this into (37),

$$\ddot{H}_u = u[\lambda_1 \cos\theta + \lambda_2 \sin\theta] = u = 0,$$
$$\forall t \in [t_1, t_2]$$

And from (36) the constant singular angle θ_S, can be determined.

$$\theta_S = \cos^{-1}(\lambda_1) = \sin^{-1}(\lambda_2)$$

Second, since our generic problem [P2] is an autonomous system, the optimality requires the 'Most Rapid Approach Path (MRAP)' [3], that is, the optimal control takes forms of 3-step picecwise continuous solution. (See [Fig4])

1. Take a full control ($u^* = 1$ or -1) until the singular angle(θ_S) is attained.
2. Take a singular control ($u^* = 0$) during the feasibility period of time.
3. Take a full control ($u^* = 1$ or -1) until the terminal conditions are satisfied. ■

Theorem 2 If $G^2 \in \Gamma^*_n$, *then* $\sigma^*_n \in \sum^*_n$ is one of the subsequences of Ω-class : (1, −

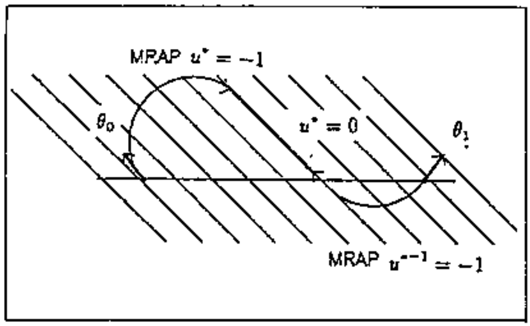


Figure 4. Most Rapid Approach Path to a singular arc

1, 1), (−1, 1, −1)

Proof : see Appendix A ■

G^2 is controlled by $\sigma^* \in \sum^*$ which is one of the subsequences among the three classes of sequences, Π, χ, and Ω. All the explicit form of extremal solutions and their related conditions are listed in Appendix B. The optimal path (G^2) is one of extremal paths described in Appendix B(6 types of solutions), which has the shortest length.

4. Conclusion

In this article, we explore the explicit form of a planar geoldesic (G^2) with given con straints. In solving G^2, a generic problem which is formulated by the optimal control approach is proposed. In general, a geodesic has a large degree of freedom. But it is shown that there are only three possible cases of the solutions in two dimensional cases.

The result of this study can be applied to the optimal path design problems such as the design of AGV lines, railroad and parking place of automobile etc. Further studies include : (i) the investigation of the generic solution characteristics of G^2 in various parameter conditions (ii) n point, closed path problem and (ⅲ) 3 dimensional geodesic etc.

References

1. BRYSON, A.E.B. and Y.C. Ho. *Applied Optimal control*, John Wiley and Sons, 1975
2. DUBINS, L.E. "On Curves of Minimal Length with a constraint on Average Curvature, and with Prescribed Initial and Terminal Points and Tangents", *American Journal of Mathematics*, 79, (1957), 497-516.
3. KAMIEN, M.I. and N.L.SCHWARTZ *Dynamic Optimization* North Holland, 1981
4. BAZARA, M.S. and C.M. SHETTY, *Nonlinear Programming*, (1979).

APPENDIX A

Principle of Optimality : An optimal path has the property that whatever the initial conditions and control values over some initial period, the control over the remaining period mist be optimal for the remaining problem. with the state resulting from the early decisions considered as the initial condition [3].

Lemma : The middle arc length of an Ω-type geodesic is always greater than π.

Proof : Generally, we can always divide the Ω-type candidates into two sorts of paths, whether the length of the middle arc is greater than π or not. Now we'll prove

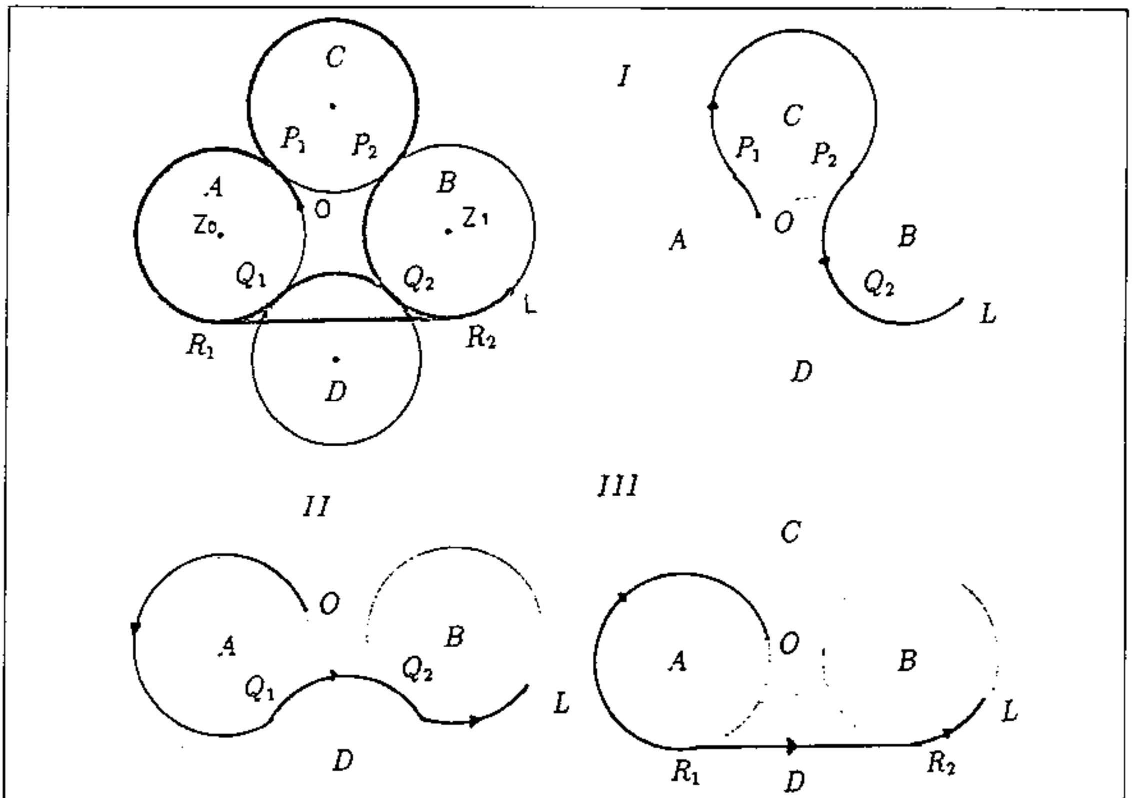


Figure 5. Proof of Lemma

that the Ω-type candidates which have their middle arc length less than or equal to π cannot be a geodesic. In Figure 5, each path of I and II represents the Ω-type extremal candidate, each has its middle arc length greater than π and less than or equal to π, respectively. Obviously, the sum of the two middle arc lengths is exactly 2π. In case of path II, there always exist a corresponding II-type candidate (path III) and we can show that path II is always dominated by path III in its length. □

Proof of Theorem 2 : In proving theorem 2, it must be shown that any extremal candidates which have 4 or more control sequences cannot be a geodesic. By priniciple of optimality, however showing that any extremal candidates which have exactly 4 control sequences cannot be a geodesic is sufficient.

Let Γ_σ be collection of all paths from A to B which characterized by σ, From the symmetricity, we only consider $\sigma = (1-1-1)$. Let, $\gamma \in \Gamma_\sigma$, $\gamma = AP_1P_2P_3B$. Let, C_i be the circle with center R_i which characterized by σ_i for $i = 1\sim4$. and C_i is tangent to C_{i+1} at the point P_i for $i = 1\sim3$. and A is on C_1, B is on C_4. Recall, C_1, C_4 are uniquely determined by Boundary condition on X, θ (see Appendix B). But C_2, C_3 are not. Thus. C_1, C_4, P_0, P_4 is fixed where as C_2, C_3, P_1, P_2,

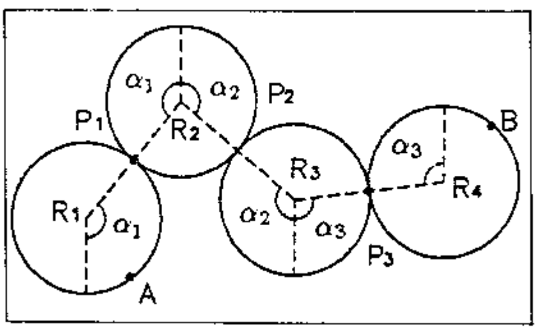


Figure 6. Example

P_3 vary with $\gamma \in \Gamma_\sigma$

Without loss of generality, Assume $C_1 = (0, 0)$, $C_4 = (d, 0)$, $P_0 = (0, -1)$, $P_4 = (d, 1)$ Next we shall show that $\gamma \in \Gamma_\sigma$ whose length is minimum cannot be Geodesic. Let, l_1, l_2, l_3, l_4 be length of AP_1, P_1P_2, P_2P_3, P_3B, respectly. Then $l(\gamma) = l_1 + l_2 + l_3 + l_4 = (l_1 - \alpha_1) + (l_2 - \alpha_2) + 2(\alpha_1 + \alpha_2 + \alpha_3)$. Note that $l_1 - \alpha_1$, $l_2 - \alpha_2$ are constants independent of $\gamma \in \Gamma_\sigma$. If $\gamma \in \Gamma_\sigma$ than $(\alpha_1, \alpha_2, \alpha_3)$ must satisfy below conditions

$$\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 = \frac{d}{2} \quad (39)$$

$$\cos\alpha_1 - \cos\alpha_2 + \cos\alpha_3 = 0 \quad (40)$$

In order to find minimun $l(\sigma)$, $\sigma \in \Gamma_\sigma$, we may solve next nonlinear problem.

MP : minimize of $\alpha_1 + \alpha_2 + \alpha_3$, subject to (39), (40).

Then Lagrangian of (MP) is

$$L = \xi(\sum \alpha_i) - \eta_1(\sum \sin\alpha_i - \frac{d}{2}) - \eta_2(\sum(-1)^{i-1}\cos\alpha_i)$$

From the Fritz-John necessary optimality condition [4]

$$\nabla_\alpha L = \xi^* e - \sum \eta_i^* g_i = 0, \ \exists \eta^* \quad (41)$$

where

g_i : *gradient vector of i-th constraint of (MP).*

$g_1^T = (\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)$

$g_2^T = (-\sin\alpha_1, \sin\alpha_2, -\sin\alpha_3)$

$e^T = (111)$

Case 1 : $\xi^* = 0$

(41) $\Rightarrow \exists \eta_i^*, \sum \eta_i^* g_i = 0$

$\Leftrightarrow g_1, g_2$: *linearly dependent*

$\Leftrightarrow g_1 \times g_2 = 0$

$$g_1 \times g_2 = (-\sin(\alpha_2+\alpha_3),\ \sin(\alpha_3-\alpha_1),\ \sin(\alpha_1+\alpha_2)) = 0$$
$$\Rightarrow \sin(\alpha_1+\alpha_2) = 0$$
$$\Leftrightarrow \alpha_1+\alpha_2 = k\pi,\ \forall k \in Z$$
$$(\text{but, } 0 < \alpha_1+\alpha_2 < 2\pi)$$
$$\Rightarrow \alpha_1+\alpha_2 = \pi$$

Subpath of j $AP_1P_2P_3$ is Ω-type curve whose midle arc length is π, it's not Geodestic, also path γ is not (see lemma).

Case 2 : $\xi^* > 0$ Without loss of generality, assume $\xi^* = 1$

$$(41) \Rightarrow \exists \eta_i^*,\ -\sum \eta_i^* g_i + e = 0 \qquad (42)$$
$$\Leftrightarrow g_1 \times g_2 \neq 0.\ det[e, g_1, g_2] = e\cdot(g_1 \times g_2) = 0$$

$$e\cdot(g_1 \times g_2) = -\sin(\alpha_2+\alpha_3) + \sin(\alpha_3-\alpha_1) - \sin(\alpha_1+\alpha_2)$$
$$= 2\sin$$
$$= 4\sin\left(\frac{\alpha_2+\alpha_3}{2}\right)\sin\left(\frac{\alpha_3-\alpha_1}{2}\right)\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)$$
$$= 0$$
$$\Leftrightarrow \alpha_2+\alpha_3 = 2k\pi \text{ or } \alpha_3-\alpha_1 = 2k\pi \text{ or } \alpha_1+\alpha_2 = 2k\pi$$
$$(\textit{since } 0 < \alpha_2+\alpha_3,\ \alpha_1+\alpha_2 < 2\pi,\ -\pi < \alpha_3-\alpha_1 < \pi)$$
$$\Rightarrow \alpha_1 = \alpha_3 (\alpha_2+\alpha_3 \neq 2k\pi,\ \alpha_1+\alpha_2 \neq 2k\pi) \qquad (43)$$

from (42) and $\alpha_1 = \alpha_3$, we can obtain η_1^*, η_2^* by next equation.

$$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \end{pmatrix}\begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (44)$$
$$\eta_1^* = \frac{\sin\alpha_1 + \sin\alpha_2}{\sin(\alpha_1+\alpha_2)} \qquad (45)$$
$$\eta_2^* = \frac{\cos\alpha_1 - \cos\alpha_2}{\sin(\alpha_1+\alpha_2)} \qquad (46)$$

From the Fritz-John sufficient optimality condition. ([4]).

$$\nabla_\alpha^2 L = \eta_1^*\, diag(\sin\alpha_1,\ \sin\alpha_2,\ \sin\alpha_3) \qquad (47)$$
$$+ \eta_2^* diag(\cos\alpha_1,\ -\cos\alpha_2,\ \cos\alpha_3) \qquad (48)$$

must be positive definite $\Leftrightarrow diag(\nabla_\alpha^2 L) > 0$

$$\Leftrightarrow \textit{since } \alpha_1 = \alpha_3, \textit{ from } (45)$$
$$\begin{pmatrix} \sin\alpha_1 & \cos\alpha_1 \\ \sin\alpha_2 & -\cos\alpha_2 \end{pmatrix}\begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} = \frac{1}{\sin(\alpha_1+\alpha_2)}\begin{pmatrix} 1-\cos(\alpha_1+\alpha_2) \\ 1-\cos(\alpha_1+\alpha_2) \end{pmatrix} > 0$$

Recall $\sin(\alpha_1+\alpha_2) \neq 0$ from (43)

subcase 1 : $\sin(\alpha_1+\alpha_2) > 0$

$\Rightarrow \cos(\alpha_1+\alpha_2) < 1$

$\Rightarrow 0 < \alpha_1+\alpha_2 < \pi$

$\Rightarrow$ *subpath of* γ $AP_1P_2P_3$ *is type of* Ω *whose middle arc length is less than* π, *greater than zero not geodestic,* γ, *is also (by Lemma).*

subcase 2 : $\sin(\alpha_1+\alpha_2) < 0$

$\Rightarrow \cos(\alpha_1+\alpha_2) > 1$

such α_1, α_2 are not exists. ∎

APPENDIX B

The explicit form of the solution of G^2

Notation

$O = (0, 0)$

$L = (0, l)$

$z_0(w_0)$ = *center of CCW(CW) circle at initial point.*

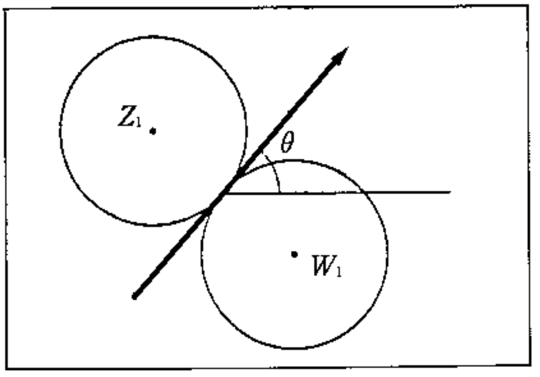
$z_1(w_1)$ = *center of CLW(CW) circle at terminal point.*

$z_0 = (-\sin\theta_0,\ \cos\theta_1)$

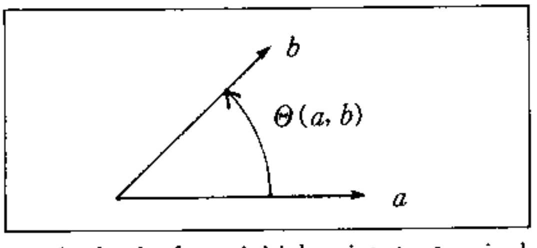
$w_0 = -z_0$

$z_1 = L + (-\sin\theta_1,\ \cos\theta_1)$

$w_1 = L - (-\sin\theta_1,\ \cos\theta_1)$



Θ(a, b) = *CCW oriented angle between a and b.*



l_i = *lenth from initial point to terminal point.*

1. Π-type Solution

(a) $\sigma = (1\ 0\ 1)$

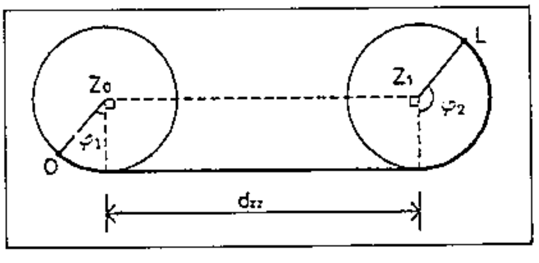


Figure 7 : Π-type (a) example

$$p = z_1 - z_0,\ d_{zz} = \|p\|$$

$$\psi_1 = [\Theta(-z_0,\ p) - \frac{\pi}{2}]\ mod\ 2\pi$$

$$\psi_2 = [\Theta(-p,\ L - z_1) - \frac{\pi}{2}]\ mod\ 2\pi$$

$$l_1 = \psi_1 + \psi_2 + d_{zz}$$

(b) $\sigma = (-1\ 0\ -1)$

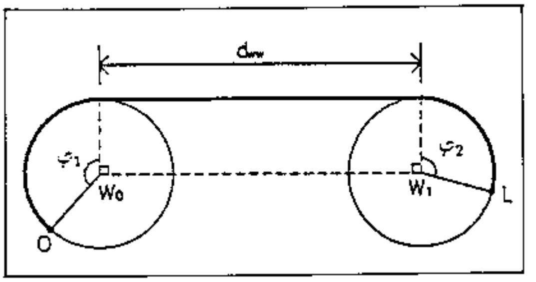


Figure 8 : Π-type (b) example

$$p = w_1 - w_0,\ d_{ww} = \|p\|$$

$$\psi_1 = [\Theta(p_1,\ w_0) - \frac{\pi}{2}]\ mod\ 2\pi$$

$$\psi_2 = [\Theta(L - w_1,\ -p) - \frac{\pi}{2}]\ mod\ 2\pi$$

$$l_2 = \psi_1 + \psi_2 + d_{ww}$$

2. X-type Solution

(a) $\sigma = (1\ 0\ -1)$

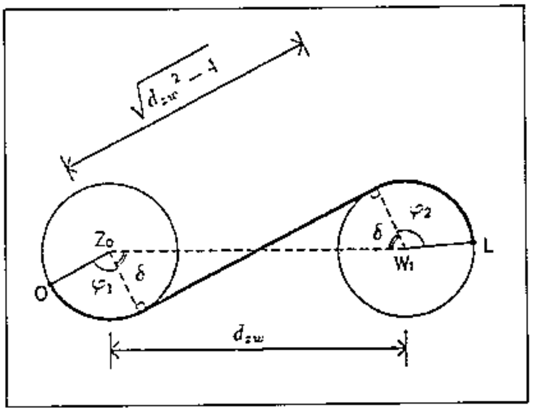


Figure 9. X-type(a) example

$$q = w_1 - z_0,\ d_{wz} = \|q\|$$

$$\delta = \arccos[\frac{2}{d_{zw}}]$$

$$\psi_1 = [\Theta(-z_0\ q) - \delta]\ mod\ 2\pi$$

$$\psi_2 = [\Theta(L - w_1,\ -q) - \delta]\ mod\ 2\pi$$

$$l_3 \quad \begin{cases} \psi_1 + \psi_2 + \sqrt{d_{zw}^2 - 4}\ if\ d_{zw} \leq 2 \\ \infty \text{ otherwise} \end{cases}$$

(b) $\sigma=(-1\ 0\ 1)$

$q\ = z_1 - w_0,\ d_{wz} = \|q\|$

$\delta\ = \arccos\ [\frac{2}{d_{zw}}]$

$\psi_1\ = [\Theta(q - w_0) - \delta]\ mod\ 2\pi$

$\psi_2\ = [\Theta(-q,\ L - z_1) - \delta]\ mod\ 2\pi$

$l_4\ = \begin{cases} \psi_1 + \psi_2 + \sqrt{d_{zw}^2 - 4}\ if\ d_{zw} \le 2 \\ \infty \text{ otherwise} \end{cases}$

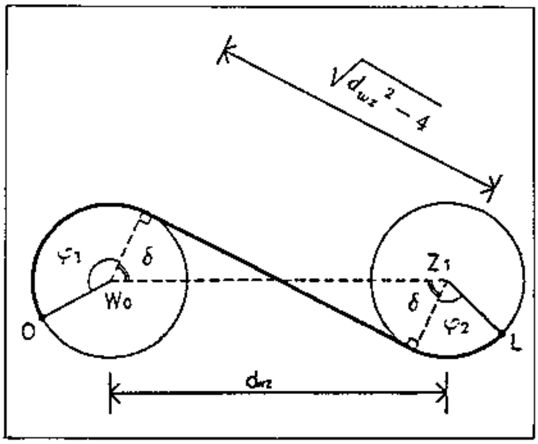


Figure 10. X-type(b) example

3. Ω-type Solution

(a) $\sigma=(1\ -1\ 1)$

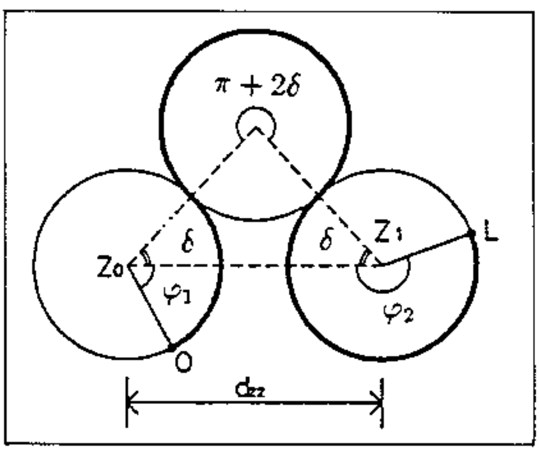


Figure 11. Ω-type(a) example

$p\ = z_1 - z_0,\ d_{zz} = \|p\|$

$\lambda\ = \arccos\ [\frac{d_{zz}}{4}]$

$\psi_1\ = [\Theta(-z_0,\ p) + \gamma]\ mod\ 2\pi$

$\psi_2\ = [\Theta(-p,\ L - z_1) + \gamma]\ mod\ 2\pi$

$l_5\ = \begin{cases} \psi_1 + \psi_2 + 2\arccos \frac{d_{zz}}{4} < 4 \\ \text{if } d_{zz} < 4 \\ \infty \text{ otherwise} \end{cases}$

(b) $\sigma=(-1\ 0\ -1)$

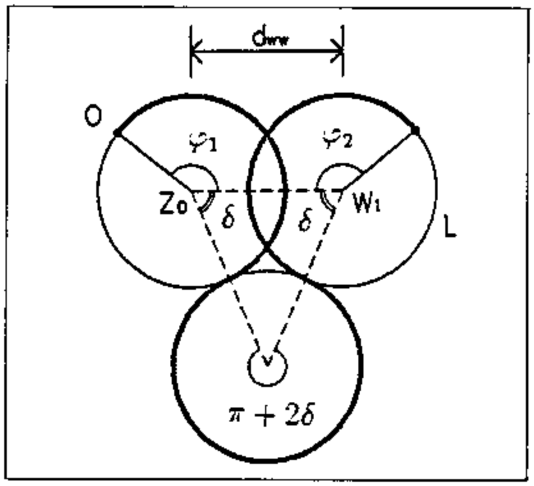


Figure 12. Ω-type(b) example

$p\ = w_1 - w_0,\ d_{ww} = \|p\|$

$\cos\gamma = [\frac{d_{ww}}{4}]$

$\psi_1\ = [\Theta(p_1, -w_0) + \gamma]\ mod\ 2\pi$

$\psi_2\ = [\Theta(L - w_1, -p) + \gamma]\ mod\ 2\pi$

$l_6\ = \begin{cases} \psi_1 + \psi_2 + \pi + \arccos\ [\frac{d_{wzz}}{4}] \\ if\ d_{wz} < 4 \\ \infty \text{ otherwise} \end{cases}$