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Adjustment of a Studentized Test Statistic and a Normalized Test Statistic in a Simple Linear Structural Relationship

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ABSTRACT

Limiting distributions of Studentized test statistics have been shown for testing the slope parameter in a simple linear structural model. Since the limiting distribution of Studentized one appears to yield inaccurate inference, this paper suggests adjustment of critical value and normalization of the Studentized one. As results, we can have procedures for refined inference based on our approximate distrbution instead of the limiting distribution.

1. Introduction

A generic simple linear measurement error model with errors in the equation is defined as:

$$(X_t\ Y_t) = (x_t\ y_t) + (e_t\ u_t),\ t = 1, \cdots, n, \qquad [1]$$

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where $y_t = (1\ x_t)\ B$ and $B^T = (\beta_0\ \beta_1)$. The vector $(X_t\ Y_t)$ is the vector of observed random variables, $(x_t\ y_t)$ is the vector of true unobserved random variables, and $(e_t\ u_t)$ is the vector of measurement errors masking the vector of true random variables.

We assume that $(x_t\ u_t\ e_t)$, $t = 1, \cdots, n$, are distributed as independent trivariate normal random vectors with mean $(\mu_x\ 0\ 0)$ and variance Diag $(\sigma_{xx}\ \sigma^2\ \sigma^2)$, where Diag means a matrix with its diagonal elements σ_{xx}, σ^2, σ^2, and its remaining elements 0. Further, all x_t's, u_t's, and e_t's are assumed to be mutually independent.

Large sample inference concerning β_1 under our model can be made using the Studentized statistic

$$T = (\hat{\beta}_1 - \beta_1) / \{\hat{V}(\hat{\beta}_1)\}^{1/2} \qquad [2]$$

where $\hat{\beta}_1$, the maximum likelihood estimator of β_1, and $\hat{V}(\hat{\beta}_1)$, a consistent estimator of the variance of the limiting distribution of $n^{1/2}\ \hat{\beta}_1$ are given by Fuller (1987):

$$\hat{\beta}_1 = [-(M_{XX} - M_{YY}) + \{(M_{XX} - M_{YY})^2 + 4 M_{XY}{}^2\}^{1/2}] / (2 M_{XY})$$

and $\hat{V}(\hat{\beta}_1) = (\hat{\sigma}^2/\hat{\sigma}_{xx})^2 + (\hat{\sigma}^2/\hat{\sigma}_{xx})(1 + \hat{\beta}_1{}^2)/(n-2) + (\hat{\sigma}^2/\hat{\sigma}_{xx})^2 \hat{\beta}_1{}^2/\{(n-1)(n-2)\}$

where $\bar{X} = \sum_{t=1}^{n} X_t / n$, $\bar{Y} = \sum_{t=1}^{n} Y_t / n$,

$$M_{XX} = \sum_{t=1}^{n} (X_t - \bar{X})^2 / (n-1),\ M_{XY} = \sum_{t=1}^{n} (X_t - \bar{X})(Y_t - \bar{Y}) / (n-1),$$

$$M_{YY} = \sum_{t=1}^{n} (Y_t - \bar{Y})^2 / (n-1),$$

$$\hat{\sigma}_{xx} = M_{XX} - \hat{\sigma}^2, \text{ and } \hat{\sigma}^2 = (M_{YY} - 2\hat{\beta}_1 M_{XY} + \hat{\beta}_1^2 M_{XX}) / (1 + \hat{\beta}_1{}^2).$$

Using Taylor series, Chang and Dahm (1993) expanded the Studentized test statistic for testing hypotheses about the slope parameter, β_1: the expansion of T is $T = t_0 + m^{-1/2} t_1 + O_p(m)$,

where $t_0 = U_{12} / \{a\sigma^2\}^{1/2}$,

$t_1 = \{U_{11} U_{12}/(3a) + U_{12} U_{22}/(3b) - U_{12}{}^2 \beta_1/c\} / \{a\sigma^2\}^{1/2} - U_{12}(aU_{22} + \sigma^2 U_{11}) / \{2(a\sigma^2)^{3/2}\}$, $(U_{11}\ U_{12}\ U_{22})^T$ is normally distributed with mean $(0, 0, 0)$ and variance-covariance matrix Diag $(2a^2, ab, 2b^2)$,

$$a = (1 + \beta_1{}^2)\sigma_{xx} + \sigma^2,$$
$$b = \sigma^2, \qquad [3]$$
$$c = a - \sigma^2,$$

and the conditional expectation of t_1 given t_0 is
$E(t_1 \mid t_0) = -\{a\sigma^2\}^{1/2}\beta_1 t_0^2 / c = B t_0^2 / c$, say. Technical details and derivation of the expressions have been relegated to Chang (1990).

They suggested the approximate distribution is more accurate approximation to the exact distributions of the Studentized statistic than is the limiting distribution. In this paper formulas are presented for computing modified critical values for tests of hypotheses about β_1 and a normalized test statistic. The former critical value is more accurate than that based on limiting distribution, and the latter normalized test statistic is compared with its critical value in standard normal distirbution table and so it is convenient.

2. Modification of critical values

Suppose that the distribution function of a statistic S is approximated by: $Pr[S \le \xi] = F(\xi) + m^{1/2} q(\xi) f(\xi) + O(m)$
where F and f are, respectively, the distribution function and the density function of a random variable whose density function satisfies

$$\frac{df}{d\xi} = h(\xi) f(\xi)$$

for a function of ξ, $h(\xi)$. Then, the value of ξ satisfying

$$Pr[S \le \xi] = 1 - \alpha,\ 0 < \alpha < 1$$

is given by

$$\xi = \xi(1-\alpha) - m^{-1/2} q(\xi(1-\alpha)) + O(m) \qquad [4]$$

where $\xi(1-\alpha)$ is the $(1-\alpha)$ percentile point of $F(\xi)$ and $m = n-2$ (Tsukuda, 1985).

For the case of our structural model the value of a level α test may be approximated to $O(m)$ as

$$T = Z(1-\alpha) - m^{-1/2} q(Z(1-\alpha)) + O(m)$$

where

$$q(Z(1-\alpha)) = (a^{1/2}\sigma\beta_1 / c)\{Z(1-\alpha)\}^2$$
$$a = (1+\beta_1^2)\sigma_{xx} + \sigma^2,\ c = a - \sigma^2,$$

since the model satisfies the supposition of the formula.

For example, when $n = 20$, $\sigma_{xx} = 4$, $\beta_1 = 0.5$, and $\sigma = 1$, the critical value of a level 0. 95 test is given by [4]:

$$T_{0.95} = Z_{0.95} - (20-2)^{-1/2} q(Z_{0.95}) = 1.49$$

where $a = (1 + (0.5)^{2})4 + 1 = 6$, $c = 6 - 1 = 5$,

$q(Z(0.95)) = ((6)^{1/2}(1)(0.5)/5)\{Z(0.95)\}^{2}$, $Z(0.95) = 1.645$

That is, adjusted portion $(20-2)^{-1/2}q(Z_{0.95}) = 1.645 - 1.49 = 0.155$ is obtained.

3. Normalization of test statistics

Let T_n be a random variable whose distribution depends on the parameters n and $\Theta = (\theta_1, \cdots, \theta_k)$. We assume that there exist $\mu(\theta)$ and $\sigma(\theta)$ such that the distribution function of $n^{1/2}(T_n - \mu(\theta)) / \sigma(\theta)$ tends to a standard normal distribution function as n goes to infinity. The parameter $\mu = \mu(\theta)$ and $\sigma = \sigma(\theta)$ may or may not depend on n. Let $f(T_n)$ be a one-to-one and twice continuously differentiable function in a neighborhood of $T_n = \mu(\theta)$. It is assumed that we can obtain an asymptotic expansion for the distribution of $f(T_n)$ having the form

$$Pr(\{f(T_n) - f(\mu) - c^*/n\} / \{\sigma f'(\mu)/n^{1/2}\} \leq x)$$
$$= \Phi(x) - n^{-1/2}(a_1\{\theta, f'(\mu), f''(\mu)\} - c^*/\{\sigma f'(\mu)\} + a_2\{\theta, f'(\mu), f''(\mu)\}x^2)\varphi(x)$$
$$+ O(n),$$

where Φ and φ are the standard normal distribution function and the standard normal probability density function, respectively, and c^* is a correction factor depending on the parameters. Let $f_0(T_n)$ be a solution to the differential equation $a_2\{\theta, f'(\mu), f''(\mu)\} = 0$. Then, taking

$$c^* = \sigma f'(\mu) a_1 \{\theta, f'(\mu), f''(\mu)\},$$

we obtain

$$Pr(\{f_0(T_n) - f_0(\mu) - c^*/n\} / \{\sigma f'(0)/n^{1/2}\} \leq x) = \Phi(x) + O(n),$$

which is due to Konishi (1981). When $E[t_1 | t_0] = A + Bt_0$ for the given expansion $T = t_0 + m^{-1/2}t_1 + O_p(m)$, it is shown that

$$f_0(x) = f(x) = e^{-2Bx}, T_n = m^{-1/2}T$$

and $c^* = f'(0)A$.

For the case of our structural model some function of T, say Z_n, which has the standard normal distribution to $O(m)$, is given by :

$$Pr[Z_n \leq \xi] = \Phi(\xi) + O(n), \qquad [5]$$

where $Z_n = \{\exp(-2m^{-1/2}BT) - 1\} / (-2m^{-1/2}B)$,

$A = 0,\ B = -\{a\sigma^2\}^{1/2}\beta_1 t_0^2 / c,$

$m = n-2,\ \mu = 0.$

For example, when $n = 20$, $\sigma_{xx} = 4$, $\beta_1 = 0.5$, and $\sigma = 1$, the normalized test statistic Z_n is given by:

$$Z_n = \{\exp(-2m^{-1/2}BT) - 1\} / (-2m^{-1/2}B),$$

$$= \{\exp(0.1154\,T) - 1\} / (0.1154).$$

If $T = 1.583$, then $Z_n = 1.74$ is obtained.

In Table 1 and 2 headings (2), (3), and (4), respectively, list empirical probabilities, standard normal probabilities and differences between numbers given in columns (2) and (3). Numbers in parentheses given in column (2) present $2X$ standard deviations of empirical probabilities. Tables 1 and 2 show that Z_n is closer to standard normal distribution $N(0, 1)$

Table 1. The empirical distribution function of T
($n = 20$, $\sigma_{xx} = 4$, $\beta_1 = 0.5$, $\sigma = 1$)

(1) ξ	(2) Empirical Pr	(3) $N(0, 1)$ Pr	(4) Errors: Col. (2) − (3)
−4	0. 002 (0. 001)	0. 000	0. 002
−3	0. 010 (0. 002)	0. 001	0. 009
−2	0. 046 (0. 004)	0. 023	0. 023
−1	0. 182 (0. 008)	0. 159	0. 023
0	0. 496 (0. 010)	0. 500	−0. 004
1	0. 848 (0. 007)	0. 841	0. 007
2	0. 982 (0. 003)	0. 977	0. 005

Table 2. The empirical distribution function of Z_n
($n = 20$, $\sigma_{xx} = 4$, $\beta_1 = 0.5$, $\sigma = 1$)

(1) ξ	(2) Empirical Pr	(3) $N(0, 1)$ Pr	(4) Errors: Col. (2) − (3)
−4	0. 000 (0. 000)	0. 000	0. 000
−3	0. 003 (0. 001)	0. 001	0. 002
−2	0. 031 (0. 003)	0. 023	0. 008
−1	0. 166 (0. 007)	0. 159	0. 007
0	0. 496 (0. 010)	0. 500	−0. 004
1	0. 834 (0. 007)	0. 841	−0. 007
2	0. 969 (0. 001)	0. 977	−0. 008

and better approximated by $N(0,\ 1)$ than is T. The empirical distributions of Z_n and T is based on randomly generated 10,000 values, respectively.

4. Conclusions

This paper deals with adjustment of critical value of the studentized test statistic of the slope parameter with known error variance ratio in simple linear structural model, and also suggests some test statistic which is normalized from the studentized one. The advantage is found from an additional adjusted term of the formulas, their examples, and simulated random generation of the studentized test statistic and the normalized one. Some extentions of these results can be attempted: (i) Consideration of the simple linear structural model with replicated observations X_{tj}, $t=1, \cdots, n$; $j=1, \cdots r_t$ and Y_{tk}, $t=1, \cdots, n$; $k=1, \cdots, s_t$; (ii) In many experiments it is unrealistic to assume that measurement errors are uncorrelated. Refined inference procedures can be developed for cases where $\sigma_{eu} = \mathrm{Cov}(u_{tj}, e_{tj}) \neq 0$.

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