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Optimal k Value for the Profit Maximizing in the k out of n : open & close Systems.

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ABSTRACT

This Paper shows a special case of the optimization criterion is to make the maximum profit in the system reliability of the k out of n open & close structure.

Especially, the number of the optimal k is determined for the profit maximization in system reliability by deriving several properties of the optimal k out of n systems in one of four possible styles (closed & opened).

1. Introduction

The system consists of n components (iid) that can be with a pre-specified frequency in one of the four possible styles. The components are subject to failures in each style. Thus, the four styles of component and system are

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1) succeeding to close
2) failure to close
3) succeeding to open
4) failure to open

that is, the system is good to close if more than k components close when closed and also is good to open if more than k components of n open when opened. In case of failure, the system is failure to close if fewer that k components close when closed and is failure to open if fewer than k components of n open when trying to open [5]. A characterization of the optimal k which maximizes the mean system-profit is obtained. This characterization is then applied to identify the situations under prediction, based on the parameters of the system, whether the optimal k is smaller than or larger than $n/2$. The same characterization also predicts the effect on the optimal k of a change in the costs of the four styles failure.

So, this paper shows to analyze and decide the optimal number of k which maximizes the mean profit of the k out of n systems.

2. Notation

I_1 : The probability of unit in succeeding to close

I_2 : The probability of unit in failing to close

I_3 : The probability of unit in succeeding to open

I_4 : The probability of unit in failing to open

n : number of units in the system

$R_1(k)$: The probability of system in succeeding to close

$R_2(k)$: The probability of system in failing to close

$R_3(k)$: The probability of system in succeeding to open

$R_4(k)$: The probability of system in failing to open

a : The probability of the closed system

$1-a$: The probability of the opened system

A^1 : The same gained value from system success in close

A^2 : The same gained value from system failure in close

A^3 : The same gained value from system success in open

A^4 : The same gained value from system failure in open

3. The criterion and properties of the maximization for k

The maximization of :

$$Z(k) = R_1(k) - R_4(k)\{(1-a)(A^3-A^4)/a(A^1-A^2)\} \qquad (1)$$

The optimization is straightforward to identify the results corresponding to this special case. A k is optimal if and only if it satisfies [1], [2].

$$Z(k) - Z(k-1) \geq 0, \quad \text{and} \quad Z(k) - Z(k+1) \geq 0 \quad \text{where} \quad n > k > 0, \qquad (2)$$

with at least one strict inequality
For $k^* = 0$ or n, $k = 0$ is optimal value if and only if

$$Z(0) \geq Z(1) \qquad (3)$$

and $k = 1$ is also optimal if and only if

$$Z(n) \geq Z(n-1) \qquad (4)$$

From the equation (2) and (4), We can get the necessary and sufficient conditions for the determination of optimal k are :

for $1 \leq k^* \leq n-1$

$$(I_2/I_4)^{n-k^*+1}(I_1/I_3)^{k^*-1} \leq (1-a)(A^3-A^4)/a(A^1-A^2)$$
$$\leq (I_2/I_4)^{n-k^*}(I_1/I_3)^{k^*+1} \qquad (5)$$

for $k^* = 0$,

$$(I_2/I_4)^n \geq (1-a)(A^3-A^4)/a(A^1-A^2) \qquad (6)$$

and for $k^* = n$,

$$(I_2/I_4)(I_1/I_3)^{n-1} \leq (1-a)(A^3-A^4)/a(A^1-A^2) \qquad (7)$$

One of the uses of equation (5), (6), (7), is that it can help predict certain properties of the magnitude of k^*, based directly on the values of the parameters $(1-a)(A^3-A^4)/a(A^1-A^2)$,

I_1, I_3. In particular, equation (5), (6), (7), delineates sufficient conditions under which k^* is :

$$k^* < n/2 + 1 \text{ ; if } (1-a)(A^3 - A^4)/a(A^1 - A^2) < 1, \text{ and } I_3 \le I_2 \quad (8)$$

$$k^* < n/2 \text{ ; if } (1-a)(A^3 - A^4)/a(A^1 - A^2) > 1, \text{ and } I_3 \ge I_2 \quad (9)$$

$$k^* = {}^1/_2(n+1), \text{ for even } n \text{ ; if } I_3 = I_2 \text{ and } (1-a)(A^3 - A^4)/a(A^1 - A^2) = 1, \quad (10)$$

$$k^* = {}^1/_2(n+1), \text{ for odd } n \text{ ; } k^* = n/2 \text{ or } n/2 + 1 \quad (11)$$

⟨ Proof of the Eq (8), (9), (10), (11) ⟩

We can rewrite Eq(5) as :

$$(n-k^*+1)\log\frac{I_1 I_2}{I_3 I_4} + (2k^*-n-2)\log\left(\frac{I_1}{I_3}\right)$$

$$\le \log[(1-a)(A^3-A^4)/a(A^1-A^2)]$$

$$\le (n-k^*)\log\left(\frac{I_1 I_2}{I_3 I_4}\right) + (2k^*-n)\log\left(\frac{I_1}{I_3}\right) \quad (12)$$

The right hand side of Eq (12) is nonnegative

because $\frac{I_1 I_2}{I_3 I_4} \ge 1$, $n-k^*+1 > 0$, $\frac{I_1}{I_3} > 1$, and $(2k^*-n-2) \ge 0$

On the other hand, $\log[(1-a)(A^3-A^4)/a(A^1-A^2)] < 0$.

Thus Eq (12) is contradicted.

Considering the case where $\{(1-a)(A^3-A^4)/a(A^1-A^2)\} = 1$, and $I_3 = I_2$

then $\frac{I_1 I_2}{I_3 I_4} = 1$ from $\frac{I_1 I_2}{I_3 I_4} \gtrless 1$, if $I_3 \lessgtr I_2$.

Thus Eq (12) becomes

$$(2k^*-n)\log[I_1/I_3] \geq 0 \geq (2k^*-n-2)\log[I_1/I_3] \qquad (13)$$

Now since $\log[I_1/I_3] > 0$, it follows from *Eq* (13) that $k^* = (n+1)/2$ if n is add. If n is even, then *Eq* (13) is satisfied for either $k^* = n/2$, or $k^* = (n/2)+1$

Theorm 1.

An interior k^* is nondecreasing in $(1-a)(A^3-A^4)/a(A^1-A^2)$.
If the increase in $(1-a)(A^3-A^4)/a(A^1-A^2)$ is sufficiently large, then an interior k^* must increase.

〈 Proof 〉

From the *Eq* (5), We can rewrite as

$$[(I_1I_4)/(I_3I_2)]^{k^*-1} \leq [(1-a)(A^3-A^4)/a(A^1-A^2)]/(I_2/I_4)^n$$
$$\leq [(I_1I_4)/(I_3I_2)]^{k^*} \qquad (14)$$

where $(I_1I_4)/(I_3I_2) > 1$ because $I_1 > I_3$. Now Let's suppose $V = (1-a)(A^3-A^4)/a(A^1-A^2)$ is changed $\bar{V}$ such that $\bar{V} > V$. If the corresponding optimal k is denoted $\bar{k}^*$, then

$$[(I_1I_4)/(I_3I_2)]^{k-1} \leq V/(I_2/I_4)^n \leq [(I_1I_4)/(I_3I_2)]^k \qquad (15)$$

Because of $\bar{V} > V$, the second part of the inequality (14) yields :
$V/(I_2I_4) > [(I_1I_4)/(I_3I_2)]^{\bar{k}^*-1}$ The preceding equation implies that the first part of the inequality (15) will be contradicted if $\bar{k}^* \leq \bar{k}^*-1$. Therefore $\bar{k}^* \geq k^*$ and if $\bar{V}$ is sufficiently larger than V, then it is obvious that *Eq* (15) will be satisfied only if $\bar{k}^* > k^*$.

4. Evaluation at k^* by the result of the derivation for $Z(k)$

This part shows some effects concerning how an interior k^* is altered due to a change in the number of components n, or due to a change in the parameters I_1 and I_3.

Our main project here is to derive qualitative effects and results with respect to the direction of change in optimal k due to a change in parameters.

The derivative of $Z(k)$ with respect to k is :

$$Z(k) = R_{1k}(k) - R_{4k}(k)V \text{ (where } V = (1-a)(A^3-A^4)/a(A^1-A^2))$$

$$R_{1k}(k) = \partial R_1(k)/\partial k$$

$$R_{4k}(k) = \partial R_4(k)/\partial k \qquad (16)$$

Thus, the interior extreme paints of $Z(k)$ are those which satisfy :

$$Z_k(k) = 0$$

It's very easy to show that $Z_k(k)$ is strictly concave in k at any k which satisfies $Z_k(k) = 0$. It follows therefore that, for an interior k^*, and $Z(k) = 0$ represents the necessary and sufficient condition for optimality (since $\partial Z_k(k)/\partial k < 0$)

Now, Using the derivation for the evaluation at k^* We obtain the following expression for dk^*/dn that is

$$\frac{dk^*}{dn} = \frac{(k^*)^2 I_2 I_4 + \{n^2-(k^*)^2\} I_1 I_3}{\{k^* I_2 I_4 + (n-k^*) I_1 I_3\} 2n} \qquad (17)$$

〈 Proof 〉

Z_k in Eq (16) can be rewritten as $Z_k = R_{4k}[(R_{1k}/R_{4k}) - V]$.
(where $V = (1-a)(A^3-A^4)/a(A^1-A^2)$). Thus, the derivative dZ_k/dk, evaluated at $Z_k = 0$, is :

$$\partial Y_k/\partial k = R_{4k}\frac{\partial(R_{1k}/R_{4k})}{\partial k}$$

$$= [k^* I_2 I_4 + (n-k^*) I_1 I_3)] R_{1k}(I_1 - I_3)/(n I_1 I_3 I_2 I_4) < 0 \qquad (18)$$

therefore,

$$\partial Z_k/\partial n = -R_{1k}(I_1 - I_3)[(k^*)^2(1-I_1-I_3) + n^2 I_1 I_3]/(2n^2 I_1 I_3 I_2 I_4) \qquad (19)$$

and hence,

$$dk^*/dn = [(k^*)^2(I_2 I_4 + \{n^2-(k^*)^2\} I_1 I_3]/\{2n(k^* I_2 I_4 + (n-k^*) I_1 I_3)\} \qquad (20)$$

Eq (17) is same as Eq (20). So, the proof is concluded like as above developments.

Eq (20) yields several qualitative conclusious concerning the local change in an interior k^* as n changes.

$\frac{dk^*}{dn}$ is between 0 and 1, and if $I_3 \gtrless I_2$ then

$\frac{dk^*}{dn} \gtrless 1/2.$

The final result, the following theorem 2 ascertains the direction of local change in k^* from a change in the probabilities of unit failure, when these probabilities are the same in the four styles structure.

Theorem 2.

$$\frac{dk^*}{dI_1} \gtrless 0, \text{ if } V \lessgtr 1, \text{ and } I_3 = I_2 \qquad (21)$$

⟨ Proof ⟩

Likes as the method and development of Eq (19)

$$\partial Z_k / I_1 = -R_{1k}(n-2k^*)(I_2^2 - I_1^2)/(2I_1^2 I_2^2) \qquad (22)$$

Using the $\partial Z / \partial K < 0$ and Eq (22), it follows that

$$\frac{dk^*}{dI_1} \gtrless 0, \text{ if } \frac{k^*}{n} \lessgtr 1/2, \qquad (23)$$

(where $\frac{k^*}{n} \lessgtr 1/2$, if $V \lessgtr 1$, and $I_3 = I_2$)

Thus Eq (21) in combination with Eq (22) implies that k^*/n becomes closer to half as the probabilities of unit failure become smaller.

5. Concluded Explanation

To test the fitability of these above results, we undertook numerical simulations for a range of parameters of exact change in K^* due to changes in parameters. The changes in K^* are computed as follows. For each combination of parameters, the integer value of K^* is first calculated directly from Eq (5) (6) (7). One of the parameters is then altered, and the new integer value of K^* is similarly calculated.

The set of simulations was for each of the following 540 ($5\times9\times3\times4=540$) combinations of parameters : n = (25, 50, 75, 100, 125)

I_1 = (0.5, 0.55, 0.6, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90)

$I_3 = (I_1 - 0.05, I_1 - 0.1, I_1 - 0.2)$

V = (0.05, 0.1, 0.75, 1.5)

In each case, the value of n was increased by 20, and the value of I_1 was increased by 0.05 and the resulting increase in the integer value of k^* was conputed. For all cases for which the pre-and post-change k^* had an interior value, the change in k^* was non-negative that is, if $I_3 \geq I_2$, then $1 \leq k^* \leq 2$, and if $I_3 \leq I_2$ then $k^* \leq 1$. And the change in k^* was non-negative for $V = (1-a)(A^3-A_4)/a(A^1-A_2) \leq 1$, and nonpositive for $V = (1-a)(A^3-A^4)/a(A^1-A^2) \geq 1$.

The objective of this paper was to analyze the K which maximizes the mean system profit. We show how one can predict, based on the parameters of the system, if the k is large of smaller than $n/2$. Also the positions of change in the k resulting from changes in system parameters are ascertained.

REFERENCES

1. M. J. Phillips, 1980, "*k out of n : G systems are preferable*", IEEE Trans. Rel, Vol. R-29, pp. 135-140.
2. R. S. Bawlow, K. D. Heidtmann, 1984, "*Computing k out of n system reliability*", IEEE Trans. Rel, Vol. R-33, pp. 322-323.
3. S. P. Jain, K. Gopal, 1985, "*Recursive algorithm for reliability evaluation of k out on n : G system* ", IEEE Trans. Reliability, Vol. R-34, pp. 144-146.
4. S. Rai, A. K. Sarjs, E. V. Prasad, A. Kumar, 1987, "*Two recursive algorithms for computing the reliability of k out on n systems*", IEEE Trans. Rel, Vol. R-36, pp. 261-265.
5. Y. Ben-Dov, 1980, "*Optimal reliability design of k out of n systems subject to two kinds of failure*", J. Opl. Res. Soc., Vol. 31, pp. 743-748