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A SHORT NOTE ON PAIRWISE NORMAL SPACES

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We give sufficient conditions for a subspace A of a pairwise normal space (X, P, L) to be SC-embedded in X.

1. Introduction

Let (X, P, L) be a topological space. A subspace A of X is SC-embedded in X [4] if every real-valued P-usc and L-lsc function on A can be extended to a P-usc and L-lsc function on X.

Since E.P. Lane [4] gave a P-closed and L-closed subset of a pairwise normal space which is not SC-embedded in contradiction with [3] theorem 2.9, it seems interesting to find conditions under which the mentioned theorem holds.

Recently, a new sufficient condition, unfortunatly lacking symmetry, in this sense can be found in [2].

First, we will show that, although Theorem 2.9 of [3] is not valid in general, the procedure used in its proof can be slightly modified to obtain a correct result for the bounded case. Later, we will give symmetric conditions for a P-L-closed subset to be SC-embedded in the pairwise normal space (X, P, L).

The abbrevations "lsc" and "usc" for lower and upper semi-continuous, respectively, are used throught.

2. Extension of semi-continuous functions on pairwise normal spaces

Theorem 2.1. *Let (X, P, L) be a pairwise normal space. Let $A \subset X$ be P-closed and L-closed. Let f be a bounded-real function defined on A*

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which is a P-usc and L-lsc function. Then there exists an extension F of f to the whole of X such that F is a P-usc and L-lsc function. The extension F can be chosen so that $F : X \to [a,b]$ *with*

$$a = \inf\{F(t) : t \in X\} = \inf\{f(t) : t \in A\}$$

and

$$b = \sup\{F(t) : t \in X\} = \sup\{f(t) : t \in A\}.$$

Proof. Let n be a positive integer. For each integer $k \in Z$, let $U_k^n = \{x : f(x) \geq k/n\}$ and $L_k^n = \{x : f(x) \leq (k-1)/n\}$. Then, for every integer k, U_k^n and L_k^n are respectively P-closed and L-closed subsets of X. Also $U_k^n \cap L_k^n = \emptyset$, $\forall k \in Z$.

By [3] Theorem 2.7 (Generalization Urysohn's Lemma), for each $k = 1, 2, \cdots$, if $U_k^n \neq \emptyset$, there is a function u_k defined on X which is a P-usc and L-lsc function on X, and such that $u_k(L_k^n)0 \leq u_k(x) \leq 1/n = u_k(U_k^n), \forall x \in X$.

If $U_k^n = \emptyset$, choose $u_k(x) = 0$, $\forall x \in X$. Also, for each $k = 0, -1, -2, \cdots$ if $L_k^n \neq \emptyset$, there is a function v_k defined on X which is a P-usc and L-lsc function on X and such that $v_k(L_k^n) = -1/n \leq v_k(x) \leq 0 = v_k(U_k^n)$, $x \in X$.

If $L_k^n = \emptyset$, choose $v_k(x) = 0, \forall x \in X$. Since f is bounded, there exists $k_n \in Z^+$ such that $U_k^n = \emptyset = L_{-k}^n$, $\forall k \geq k_n$, $k \in Z$. Therefore, $\forall k \geq k_n$ we have $u_k(x) = v_{-k}(x) = 0$, $\forall x \in X$, and so

$$f_n = \sum_{k=1}^{\infty} u_k + \sum_{k=0}^{\infty} v_{-k}$$

is a functional series with a finite number of non-zero terms. Now, it is obvious that

$$f_n(x) = \sum_{k=1}^{\infty} u_k(x) + \sum_{k=0}^{\infty} v_{-k}(x)$$

is a bounded-real function defined on X, which is a P-usc and L-lsc function for each $n \in Z^+$.

In [3] it is shown that the restrictions $f_n|_A (n = 1, 2, \cdots)$, converge uniformely to f on A, and form a Cauchy sequence. By Theorem 3.6 of [3], f has an extension F to X which satisfies the theorem conditions.

Counterexample 2.2.

The following example (originally due to Lane [4]) clearly illustrates how the procedure used in the proof of Th.2.9 of [3] can fail when the initial function f is not bounded. As it can be easily seen, the main reason for this failure consists in the non-guaranteed convergence of the above defined series:

Let X be an uncountable set. Let P be the *co*-countable topology on X, and L the discrete topology. (X, P, L) is pairwise normal. Consider the countably infinite subset $A = \{x_1, x_2, \cdots\}$ of X. Then, $f(x_1) = i(i = 1, 2, \cdots)$ is a P-lsc and L-usc function on the P-L-closed subset A of X, without extension P-lsc and L-usc to X.

Under the notation of the theorem 2.1, let

$$f_n(x) = \sum_{k=1}^{\infty} u_k(x) + \sum_{k=0}^{\infty} v_{-k}(x), x \in X$$

For each integer $k \geq 1$ we have:

$$\begin{aligned} X &= [\cup_{j=1}^{\infty} u_k^{-1}(] - \infty, \frac{1}{n} - \frac{1}{j}]) \cup u_k^{-1}(1/n) \\ &= [\cup_{j=1}^{\infty} V_{-k}^{-1}(] - \infty, -1/j])] \cup v_{-k}^{-1}(0) \end{aligned}$$

Since u_k and v_{-k} are P-lsc functions on X, for $k \geq 1$, then the subsets of X $u_k^{-1}(] - \infty, \frac{1}{n} - \frac{1}{j}])$ and $v_k^{-1}(] - \infty, -1/j])$ are P-closed and therefore they are countables. Consequently, $u_k^{-1}(1/n)$ and $v_{-k}^{-1}(0)$ are non-countable subsets of X and therefore the subset of X

$$B = \cap_{k=1}^{\infty}[u_k^{-1}(1/n) \cap v_{-k}^{-1}(0)]$$

has countable complement in X, and for each $x \in B$ the series

$$f_n(x) = \sum_{k=1}^{\infty} u_k(x) + \sum_{k=0}^{\infty} v_{-k}(x) = (\sum_{n=1}^{\infty} 1/n) + v_0(x)$$

is not convergent.

Lane's example shows the problem of SC-embedding can be a problem relative to one topology only. Finally, it is easy to extend Lane's example as follows:

Let X be an uncountable set; suppose (X, P) is an space of Second Category and suppose all the proper P-closed subsets are nowhere dense in X. Let L be a topology on X such that (X, P, L) is pairwise normal.

If A is an infinite countable P-closed T_1 subsepace of X, then A is not SC-embedded in X.

3. SC-embedded subsets

Theorem 3.1. *Let (X, P, L) be pairwise normal. Each P-closed and L-closed subset A of X such that the P-boundary $b_p(A)$ is L-countably compact and the L-boundary $b_L(A)$ is P-countably compact, is SC-embedded in X.*

Proof. Let f be a P-usc and L-lsc function on the P-closed and L-closed subset A of X.

At first, we suppose the boundaries $b_p(A)$ and $b_L(A)$ are non-empty. Then, f is a L-lsc function on $b_p(A)$ which is L-countably compact and from [1] Prop. 3.17, f has a lower bound m on $b_p(A)$. In the same way f has an upper bound M on $b_L(A)$ and we can suppose $M > m$.

The function $f_1(x) = f(x)$, $x \in A$ and $f_1(x) = m, x \in X - A$, is P-usc function on X; in fact:

If $a > m$, then $f_1^{-1}([a, +\infty[)$ is $P|_A$-closed and therefore, P-closed.

If $a \leq m$, then $f_1^{-1}([a, +\infty[= (X - A) \cup f^{-1}([a, \infty[)$ is P-closed since $b_p(A) = b_p(X - A) \subset f^{-1}([a, \infty[)$.

In the same way, the function $f_2(x) = f(x)$, $x \in A$ and $f_2(x) = M$, $x \in X - A$, is L-lsc function on X.

Therefore, we have $f_1 \leq f_2$ on X, and from [4], Th. 2.5, there exists a P-usc and L-lsc function h on X such that $f_1 \leq h \leq f_2$, but $f_1(x) = f_2(x) = f(x)$ $x \in A$, and therefore f is the required extension.

Suppose now that $b_p(A) = \emptyset$. Then $X - A$ is P-closed and the above function f_1 is a P-usc function on X, for each $m \in R$. Also, if $b_L(A) = \emptyset$, then the above function f_2 is a L-lsc function on X, for each $M \in R$. In all the possible cases, it suffices to choose $m \leq M$, and to repeat the argument of last paragraph.

The following characterization of pairwise countable compactness [5], is due to Singal and Singal [6]: A bitopological space (X, P, L) is pairwise countably compact if and only if every proper $P(L)$-closed subset of X is $L(P)$-countably compact.

Since countably compact property is closed-hereditary, we have as an inmediate consequence the following

Corollary 3.2. *Every P-closed and L-closed subset of a pairwise countably compact space (X, P, L) is SC-embedded.*

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References

[1] P. Fletcher, W.F. Lindgren, *Quasi-uniform spaces*, Marcel Dekker, Inc. New York, 1982.

[2] A. Gutiérrez, S. Romaguera, *A postscript to "C-Binary Relations and pairwise normality"*, An. St. Univ. Al. I. Cuza Iasi Matematica, 35 (1989).

[3] J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. (3) 13 (1963), 71-89.

[4] E.P. Lane, *Bitopological spaces and quasi-uniform spaces*, Proc. London. Math. Soc. (3) 17, (1967) 241-256.

[5] D.H. Pahk, B.D. Choi, *Notes on pairwise compactness*, Kyungpook Math.J. 11(1971), 45-52.

[6] M.K. Singal, R. Singal, *Some more separation Axioms in Bitopological spaces*, Ann. Soc. Sci. Bruxelles 84 (1970), 207-230.

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