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A GENERALIZATION OF PERRON'S STABILITY THEOREMS FOR PERTURBED LINEAR DIFFERENTIAL EQUATIONS

Sung Kag Chang*, Hyung Jae Lee and Young Sun Oh

In this paper, sharp stability properties are obtained for perturbed linear differential equations by generalizing Perron's type stability theorems, and some examples of them are given.

1. Introduction

Recently, a number of authors —F.M. Dannan, H.R. Farran, A. Halanay, J.K. Hale, P. Hartman, T. Taniguchi, S.K. Chang, etc.— have studied the stability theory of differential equations, and also some of them have tried to generalize Perron's celebrated theorem.

In such qualitative theory, Lyapunov's and Perron's stability theorems are most important and popular in the literatures [1, 8, 9, 10, 12], etc.

It is an interesting problem under what conditions for the perturbations the qualitative properties of the original equations are preserved or improved in a suitable sense.

In this paper, we shall define notions of stability, which are called $\varphi(t)$-(uniform) stability $\varphi(t)$-quasi-(uniform) asymptotic stability and $\varphi(t)$-(uniform) asymptotic stability of the solutions for systems of differential equations. Then we are concerned with generalization of Perron's type celebrated stability theorems by using $\varphi(t)$-stability concepts for a positive real function $\varphi(t)$ on R^+ and we also obtain the results in [5, 12] as corollaries of our results. In section 2, we discuss the equivalent conditions of $\varphi(t)$-stability and $\varphi(t)$-uniform stability for the solutions of linear differential equations, in section 3 we discuss $\varphi(t)$-stability for the solutions

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of the perturbed differential equation $\frac{dx}{dt} = A(t)x + f(t,x)$, and also we give some examples, to which our results can be applied.

2. Preliminaries

Throughout this paper, let R^n be the n-dimensional Euclidean space and $R^+ = [0, \infty)$.

For a given function $g(t,y) \in C[R^+ \times R^n, R^n]$, we consider following differential equations:

$$\frac{dy}{dt} = g(t,y) \tag{2.1}$$

Let us assume that $g(t,0) = 0$ for all $t \in R^+$, and the equation (2.1) is well-posed for sufficiently small initial values at any initial time.

$y(t;t_1,y_1)$ denotes by the solution of (2.1) with initial condition $y(t_1) = y_1$, $y_1 \in R^n$, $t_1 \geq 0$.

Now we introduce generalized definitions of stability for the equation (2.1):

Definition 2.1. Let $\varphi(t)$ be a positive real function on R^+. The zero solution of (2.1) is said to be *$\varphi(t)$-stable*[$\varphi(t)$-S] if for any $\varepsilon > 0$ and for any $t_1 \geq 0$, there exists $\delta(t_1,\varepsilon) > 0$ such that if $\|y(t_1)\| < \delta(t_1,\varepsilon)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1$; *$\varphi(t)$-uniformly stable*[$\varphi(t)$-US] if the $\delta(t_1,\varepsilon)$ in [$\varphi(t)$-S] is independent of time t_1; *$\varphi(t)$-quasi-asymptotically stable*[$\varphi(t)$-QAS] if for any $\varepsilon > 0$ and for any $t_1 \geq 0$, there exist $\delta(t_1) > 0$ and $T(t_1,\varepsilon) > 0$ such that if $\|y(t_1)\| < \delta(t_1)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1,\varepsilon)$; *$\varphi(t)$-quasi-uniformly asymptotically stable*[$\varphi(t)$-$QUAS$] if the $\delta(t_1)$ and the $T(t_1,\varepsilon)$ in [$\varphi(t)$-QAS] are independent of time t_1; *$\varphi(t)$-asymptotically stable*[$\varphi(t)$-AS] if it is $\varphi(t)$-stable and $\varphi(t)$-quasi-asymptotically stable; *$\varphi(t)$-uniformly asymptotically stable*[$\varphi(t)$-UAS] if it is $\varphi(t)$-uniformly stable and $\varphi(t)$-quasi-uniformly asymptotically stable.

In particular, if we put $\varphi(t) = ke^{nt}$ where n is a real number and k is a constant, then $\varphi(t)$-stability, etc. will be $T(n)$-stability, etc., respectively (cf.[5], [12]). Also, the $T(n)$-stability concepts are exactly the same as the usual definitions of stability when $n = 0$.

Now we present a modifed lemma for integral inequalities.

Lemma 2.2. ([9], p.315) *Assume that the following conditions hold for functions $f(t), g(t) \in C[[t_1,\infty), R^+]$ and $F(t,s,u)$, $t \geq s \geq t_1$, $u \geq 0$:*
$f(t) - \int_{t_1}^{t} F(t,s,f(s))ds < g(t) - \int_{t_1}^{t} F(t,s,g(s))ds$, $t \geq t_1$ and $F(t,s,u)$

is monotone nondecreasing in u *for each fixed* $t \geq s \geq 0$.

Then we have that $f(t) \leq g(t)$ *for all* $t \geq t_1$, *when* $f(t_1) < g(t_1)$.

Let $A(t)$ be a continuous $n \times n$ matrix defined on R^+ and let $f(t,x) \in C[[0,\infty) \times R^n, R^n]$ with $f(t,0) = 0$ for any $t \in [0,\infty)$.

Consider a linear differential equation

$$\frac{dx}{dt} = A(t)x \tag{2.2}$$

and a perturbed differential equation of the above:

$$\frac{dx}{dt} = A(t)x + f(t,x). \tag{2.3}$$

Let $U(t)$ be a fundamental matrix of (2.1). Then it is well known that the solution $x(t)$ of (2.3) satisfies the integral equation

$$x(t) = U(t)U^{-1}(t_1)x(t_1) + \int_{t_1}^{t} U(t)U^{-1}(s)f(s,x(s))ds, \ t \geq t_1. \tag{2.4}$$

Next, we obtain the following basic lemma.

Lemma 2.3. *Let* $k(t)$ *be a positive real function on* R^+. *Then the zero solution of the differential equation* (2.2) *is* $k(t)$*-stable if and only if* $\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1}$, $t \geq s \geq 0$ *for some positive real function* $h(t)$ *on* R^+.

Proof. Let $x(t) \equiv x(t;t_1,x_1)$ be a solution of (2.2) with an initial value (t_1,x_1), $t_1 \geq 0$. Then we have the expression $x(t) \equiv U(t)U^{-1}(t_1)x_1$, $t \geq t_1 \geq 0$. Suppose that the zero solution of (2.2) is $k(t)$-stable for a positive real function $k(t)$ on R^+. Then for $\varepsilon = 1$, there exists a positive real number $\delta(t_1)$ such that if $\|x(t_1)\| < \delta(t_1)$, then $\|x(t)k(t)^{-1}\| < 1$ for all $t \geq t_1 \geq 0$. Let $u_1 \in R^n$ be an arbitrary vector with $\|u_1\| < 1$. Then $\|\delta(t_1)u_1\| < \delta(t_1)$ and hence $\|U(t)U^{-1}(t_1)\delta(t_1)u_1k(t)^{-1}\| < 1$, $t \geq t_1 \geq 0$. The relation $\|U(t)U^{-1}(t_1)\delta(t_1)u_1k(t)^{-1}\| = \delta(t_1)k(t)^{-1}\|U(t)U^{-1}(t_1)u_1\|$ yields $\|U(t)U^{-1}(t_1)u_1\| < \frac{1}{\delta(t_1)}k(t)$. Since u_1 is arbitrary, with $\|u_1\| < 1$, we have $\|U(t)U^{-1}(t_1)\| \leq \frac{1}{\delta(t_1)}k(t)$. Thus, without loss of generality, we can take continuous function $\delta(t)$ on $[0,\infty)$. Taking $h(t) = \delta(t)$, we have the result.

Conversely, suppose that there exist positive real functions $k(t)$, $h(t)$ on R^+ such that $\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1}$, $t \geq s \geq 0$. Given any $\varepsilon > 0$ and any $t_1 \geq 0$, there exists $\delta(t_1)$ with $0 < \delta(t_1) < h(t_1)\varepsilon$ such that if

$\|x(t_1)\| < \delta(t_1)$, then $\|x(t)k(t)^{-1}\| = h(t_1)^{-1}\|x(t_1)\| < h(t_1)^{-1}h(t_1)\varepsilon = \varepsilon$ for all $t \geq t_1 \geq 0$. Thus the zero solution of (2.2) is $k(t)$-stable for a positive real function $k(t)$ on R^+.

Lemma 2.4. *Let $k(t)$ be a positive real function on R^+. Then the zero solution of the differential equation (2.2) is $k(t)$-uniformly stable if and only if $\|U(t)U^{-1}(s)\| \leq rk(t),\ t \geq s \geq 0$ for some positive real number r.*

Proof. We can prove the lemma by the analogous method to the proof of lemma 2.3, taking the functions $\delta(t)$ as a constant.

3. $\varphi(t)$–Stability Theorems

In this section we discuss $\varphi(t)$-stability of the zero solution for the perturbed differential equation (2.3).

Assume that $f(t,0) \equiv 0$ for all $t \in R^+$ throughout this section.

Let us consider the following differential equations

$$\frac{dy}{dt} = h(t)^{-1}F(t, k(t)y(t)) \tag{3.1}$$

$$\frac{dy}{dt} = F(t, k(t)y(t)) \tag{3.2}$$

where $k(t)$ and $h(t)$ are continuous positive real functions on R^+.

Henceforth, we assume that the above differential equations (3.1) and (3.2) possess the existence and the uniqueness properties of solutions on R^+ for sufficiently small initial values.

We are now in a position to prove our results.

Theorem 3.1. *Let the following conditions hold for the differential equation (2.3):*

(1a) *$\|f(t,x)\| \leq F(t, \|x\|)$, $F(t,0) \equiv 0$, and $F(t,u)$ is monotone non-decreasing with respect to u for each fixed $t \geq 0$,*

(1b) *$F(t,u) \in C[[0,\infty) \times R^+, R^+]$,*

(1c) *The zero solution of the differential equation (2.1) is $k(t)$-stable for a positive real function $k(t)$ on R^+, that is, there exist functions $k(t) > 0$ and $h(t) > 0$ on R^+ such that*

$$\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1},\ t \geq s \geq 0.$$

If the zero solution of the differential equation (3.1) is $\varphi(t)$-stable [$\varphi(t)$-quasi-asymptotically stable], for a positive real function $\varphi(t)$ on R^+, then *the zero solution of (2.3) is $k(t)\varphi(t)$-stable [$k(t)\varphi(t)$-quasi-asymptotically stable].*

Proof. First, we shall prove that the zero solution of (2.3) is $k(t)\varphi(t)$-stable for a positive real function $\varphi(t)$ on R^+. Let $x(t) \equiv x(t; t_1, x_1)$ be a solution of (2.3) with an initial value (t_1, x_1), $t_1 \geq 0$. Then the solution $x(t)$ is of the form (2.4).
Thus we obtain that from conditions ($1a$) and ($1c$),

$$\|x(t)\| \leq k(t)h(t_1)^{-1}\|x_1\| + k(t)\int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds.$$

So $k(t)^{-1}\|x(t)\| \leq h(t_1)^{-1}\|x_1\| + \int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds$.
Thus let $y(t) \equiv y(t; t_1, y_1)$ be the solution of (3.1) passing through (t_1, y_1) and let $h(t_1)^{-1}\|x_1\| < y_1$. Then $k(t)^{-1}\|x(t)\| - \int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds \leq h(t_1)^{-1}\|x_1\| < y_1$. While, since $y_1 = y(t) - \int_{t_1}^{t} h(s)^{-1}F(s, k(s)y(s))ds$,

$$\begin{aligned} &\|x(t)\| - k(t)\int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds \\ < \ & k(t)y(t) - k(t)\int_{t_1}^{t} h(s)^{-1}F(s, k(s)y(s))ds. \end{aligned}$$

Therefore, applying lemma 2.2, we obtain that $\|x(t)\| < k(t)y(t)$, by taking x_1 as $\|x_1\| < k(t_1)y_1$ $t \geq t_1$. Hence the zero solution of (2.3) is $k(t)\varphi(t)$-stable for a positive real function $\varphi(t)$ on R^+.

Next, suppose that the zero solution of (3.1) is $\varphi(t)$-quasi-asymptotically stable for a postive real function $\varphi(t)$ on R^+. Then for any $\varepsilon > 0$ and any $t_1 \geq 0$, there exist $\delta_1(t_1) > 0$ and $T(t_1, \varepsilon) > 0$ such that if $\|y(t_1)\| < \delta_1(t_1)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$. Thus set $\delta(t_1) = h(t_1)\delta_1(t_1)$. If $\|x(t_1)\| < \delta(t_1)$, then we can take $y_1 > 0$ such that $h(t_1)^{-1}\|x(t_1)\| < y_1 < \delta_1(t_1)$. Accordingly, $\|x(t)\| < k(t)y(t)$ for all $t \geq t_1 + T(t_1, \varepsilon)$ and $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$. Therefore we have $\|x(t)(k(t)\varphi(t))^{-1}\| < \|k(t)y(t)k(t)^{-1}\varphi(t)^{-1}\| = \|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$, the zero solution of (2.3) is $k(t)\varphi(t)$-quasi-asymptotically stable for a positive real function $\varphi(t)$ on R^+.

We get the following from Theorem 3.1.

Corollary 3.2. ([5]) *Let the conditions* (1a), (1b), *and* (1c) *hold for the differential equation (2.3). If the zero solution of the differential equation* (3.1) *is $T(n)$-stable [$T(n)$-quasi-asymptotically stable], for a real number*

n, then the zero solution of the differential equation(2.3) is $k(t)e^{nt}$-stable [$k(t)e^{nt}$-quasi- asymptotically stable].

In particular, if $k(t) = Ke^{mt}$ (in (1c) and (3.1)) for a real number m and a positive constant K and if the zero solution of (3.1) is $T(n)$-stable [$T(n)$-quasi-asymptotically stable], for a real number n, then the zero solution of (2.3) is $T(m+n)$-stable [$T(m+n)$-quasi-asymptotically stable].

By the similar method as in the proof of theorem 3.1, we have the following.

Theorem 3.3. *Let the conditions (1a) and (1b) hold for the differential equation (2.3). Furthermore, suppose that the following condition is satisfied:*

(2c) *The zero solution of the differential equation (2.2) is $k(t)$-uniformly stable for a positive real function $k(t)$ on R^+, that is, there exists a function $k(t) > 0$ such that $\|U(t)U^{-1}(s)\| \leq k(t)$, $t \geq s \geq 0$.*

If the zero solution of the differential equation (3.2) is $\varphi(t)$-uniformly stable [$\varphi(t)$-stable, $\varphi(t)$-quasi-(uniformly) asymptotically stable], for a positive real function $\varphi(t)$ on R^+, then the zero solution of (2.3) is $k(t)\varphi(t)$-uniformly stable [$k(t)\varphi(t)$-stable, $k(t)\varphi(t)$-quasi-(uniformly) asymptotically stable].

We get the following from Theorem 3.3.

Corollary 3.4. ([5]) *Let the conditions (1a), (1b), and (2c) hold for the differential equation(2.3).*

If the zero solution of the differential equation (3.2) is $T(n)$-uniformly stable[$T(n)$-stable, $T(n)$-quasi-(uniformly) asymptotically stable], for a real number n, then the zero solution of the differential equation (2.3) is $k(t)e^{nt}$-uniformly stable [$k(t)e^{nt}$-stable, $k(t)e^{nt}$-quasi-(uniformly) asymptotically stable].

In particular, if $k(t) = Ke^{mt}$ (in (2c) and (3.2)) for a real number m and a positive constant K and if the zero solution of (3.2) is $T(n)$-uniformly stable [$T(n)$-stable, $T(n)$-quasi-(uniformly) asymptotically stable], for a real number n, then the zero solution of (2.3) is $T(m+n)$-uniformly stable [$T(m+n)$-stable, $T(m+n)$-quasi-(uniformly) asymptotically stable].

Remark. If $k(t)$ is a positive constant, then the equations (3.1) and (3.2) are equivalent.

Now we give an example for Theorem 3.1.

Example 3.5. Consider the two-dimensional perturbed linear system:

$$\begin{aligned} \dot{x} &= -\frac{1}{2\sqrt{t+1}}x + f_1(t,x,z) \\ \dot{z} &= -\frac{1}{2\sqrt{t+1}}z + f_2(t,x,z) \end{aligned} \tag{3.3}$$

where $f_1(t,x,z) = z^2$ and $f_2(t,x,z) = x^2$. Then

$$A(t) = \begin{pmatrix} -\frac{1}{2\sqrt{t+1}} & 0 \\ 0 & -\frac{1}{2\sqrt{t+1}} \end{pmatrix}$$

is a continuous 2×2 matrix defined on R^+. Consider the linear equations:

$$\begin{aligned} \dot{x} &= -\frac{1}{2\sqrt{t+1}}x \\ \dot{z} &= -\frac{1}{2\sqrt{t+1}}z \end{aligned} \tag{3.4}$$

Then $x(t) = x_1 r_1 e^{-\sqrt{t+1}}$ and $z(t) = z_1 r_1 e^{-\sqrt{t+1}}$ are the solutions of (3.4) where $r_1 = e^{+\sqrt{t_1+1}}$.

Thus a fundamental matrix for (3.4) is given by

$$U(t) = \begin{pmatrix} e^{-\sqrt{t+1}} & 0 \\ 0 & e^{-\sqrt{t+1}} \end{pmatrix}, \quad t \geq 0,$$

which has the norm: $\|U(t)\| = e^{-\sqrt{t+1}}$, $t \geq 0$. Hence it can be taken as $k(t) = h(t) = e^{-\sqrt{t+1}}$ and $F(t,u) = u^2$. Now consider the differential equation

$$\frac{dy}{dt} = h(t)^{-1}F(t, k(t)y(t)) = e^{-\sqrt{t+1}}y^2 \tag{3.5}$$

Then the solution of (3.5) is given by $y(t) = \dfrac{y_1}{1 - y_1 \int_{t_1}^t e^{-\sqrt{t+1}}dt}$, $t \geq t_1 \geq 0$.

Note that $\int_{t_1}^t e^{-\sqrt{t+1}}dt \leq \int_0^\infty e^{-\sqrt{t+1}}dt = 4e^{-1}$. Take y_1 as $|y_1| < \frac{e}{4}$. Then

$$\frac{|y_1|}{1 + |y_1|4e^{-1}} < |y(t)| = \frac{|y_1|}{|1 - y_1 \int_{t_1}^t e^{-\sqrt{t+1}}dt|} < \frac{|y_1|}{1 - |y_1|4e^{-1}}.$$

Therefore, the zero solution of $\frac{dy(t)}{dt} = e^{-\sqrt{t+1}}y^2$ is (uniformly) stable.

Hence the zero solution of the system (3.3) is $e^{-\sqrt{t+1}}$-(uniformly) stable. Moreover, the zero solution of the system is asymptotically stable, but not exponentially. This sharp stability property cannot be obtained by the results in [5, 12]

We shall give another example.

Example 3.6. Let the following conditions hold for the differential equation (2.3):

(1e) $\|f(t,x)\| \le a(t)\|x\|$,

(1f) $a(s) \in C[[0,\infty), R^+]$ and there exist a positive constant M and a positive real function $k(s)$ such that $\int_0^\infty a(s)k(s)ds < M$.

If the condition (2c) holds, then the zero solution of (2.3) is $k(t)$-uniformly stable.

Proof. Set $F(t,u) = a(t)u,\ u \ge 0$.

We show that the zero solution of the differential equation

$$\frac{dy}{dt} = a(t)k(t)y(t) \tag{3.6}$$

is uniformly stable.

In fact, first of all, let $y(t) \equiv y(t;t_1,y_1)$, $t \ge t_1 \ge 0$ be a solution of (3.6) passing through (t_1,y_1). Then we obtain that

$y(t) = y(t_1)e^{\int_{t_1}^t a(s)k(s)ds} \le y(t_1)e^M$ because $\int_0^\infty a(s)k(s) < M$.

Accordingly, the zero solution of (3.6) is uniformly stable.

Thus by theorem 3.3, the zero solution of (2.3) is $k(t)$-uniformly stable.

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DEPARTMENT OF MATHEMATICS, YEUNGNAM UNIVERSTY, GYONGSAN 713-749, KOREA.

DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, KYOUNGJU 780-719, KOREA.

DEPARTMENT OF MATHEMATICS EDUCATION, TAEGU UNIVERSITY, TAEGU 705-033, KOREA.