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ON A STRENGTH OF TWO ABSOLUTE SUMMABILITY METHODS

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A new absolute summability method is defined. A new theorem concerning this method is proved. Other results, some of them are known, are deduced.

1. Introduction

Let $\sum a_n$ be an infinite series of partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1}|\eta_n|^k < \infty.$$

Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|t_n - t_{n-1}|^k < \infty \quad (\text{Bor [1]}),$$

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where

$$t_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v.$$

The series $\sum a_n$ is said to be summable $|R, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty.$$

If we take $p_n = 1$, each of the two summabilities $|\bar{N}, p_n|_k$ and $|R, p_n|_k$ is the same as $|C, 1|_k$ summability. Let $\{\varphi_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n, \varphi_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |t_n - t_{n-1}|^k < \infty \quad \text{(Sulaiman [7])}.$$

The series $\sum a_n$ is summable $|N, p_n|$ if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty, \tag{1}$$

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v \quad (T_{-1} = 0).$$

We write $p = \{p_n\}$ and

$$M = \{p : p_n > 0 \& \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, n = 0, 1, \cdots\}.$$

It is known that for $p \in M$, (1) holds iff (Das [4])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} |\sum_{v=1}^{\infty} p_{n-v} v a_v| < \infty.$$

For $p \in M$, $\sum a_n$ is summable $|N, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} |\sum_{v=1}^{n} p_{n-v} v a_v|^k < \infty. \text{ (Sulaiman [6])}$$

In the special case in which $p_n = A_n^{r-1}, r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1 - x)^{-r-1}$ for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C, r|_k$ summability. Here we give the following new definition :

Definition. Let $p \in M$. The series $\sum a_n$ is said to be summable $|N, p_n, \varphi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} (\frac{p_n}{P_n P_{n-1}})^k |\sum_{v=1}^{n} v p_{n-v} a_v|^k < \infty.$$

The following result is due to Bosanquet [3].

Theorem A. *Let $k = 1, p_n \& q_n > 0, P_n \& Q_n \to \infty$. In order that*

$$|R, p_n|_k \Rightarrow |R, q_n|_k \tag{2}$$

it is necessary and sufficient that

$$q_n P_n / p_n Q_n = 0(1). \tag{3}$$

Bor [2], generalized the above result by giving the following.

Theorem B. *In order for (2) to hold, (3) is necessary. If we suppose that*

$$\sum_{n=v}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = 0\{\frac{v^{k-1} q_v^{k-1}}{Q_v^k}\} \tag{4}$$

then (3) is also sufficient.

We assume $\{\alpha_n\}, \{\beta_n\}$ be sequence of positive numbers and prove the following

Theorem C. *Let $q \in M$ such that $\{\alpha_n^{1-1/k} q_n / Q_n Q_{n-1}\}$ nonincreasing. Let t_n denote the $(\bar{N}, p_n)$-mean of the series $\sum a_n$ and write $\gamma_n = \beta_n^{1-1/k} \Delta t_{n-1}$. If*

$$\sum_{n=v}^{\infty} \alpha_n^{k-1} (\frac{q_n}{Q_n})^k \frac{q_{n-v-1}}{Q_{n-1}} = 0\{\alpha_v^{k-1} (\frac{P_v}{p_v})^k (\frac{q_v}{Q_v Q_{v-1}})^k\},$$

$$\sum_{n=1}^{\infty} n^k (\frac{\alpha_n}{\beta_n})^{k-1} (\frac{P_n}{p_n})^k (\frac{q_n}{Q_n Q_{n-1}})^k |\epsilon_n|^k |\gamma_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} n^k (\frac{\alpha_n}{\beta_n})^{k-1} (\frac{P_n}{p_n})^k (\frac{P_{n-1}}{p_n})^k (\frac{q_n}{Q_n Q_{n-1}})^k |\Delta \epsilon_n|^k |\gamma_n|^k < \infty,$$

and

$$\sum_{n=1}^{m} (\frac{\alpha_n}{\beta_n})^{k-1} (\frac{P_{n-1}}{p_n})^k (\frac{p_n}{p_n})^k (\frac{q_n}{Q_n Q_{n-1}})^k |\epsilon_n|^k |\gamma_n|^k,$$

then the series $\sum a_n \epsilon_n$ is summable $|N, q_n, \alpha_n|_k, k \geq 1$.

2. Lemmas

Lemma 1 (Bor [2]). *Let $k > 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (\ell^k; \ell^k)$, it is necessary that*

$$a_{nv} = 0(1) \quad (all\ n, v). \tag{5}$$

Lemma 2 (Sulaiman [8]). *If $\{q_n\}$ nonincreasing, then for $0 < r \leq 1$,*

$$\sum_{n=v}^{\infty} \frac{q_{n-v-1}}{n^r Q_n} = 0(v^{-r}).$$

3. Proof of Theorem C

Write

$$\tau_n = \sum_{v=1}^{n} v q_{n-v} a_v \epsilon_v.$$

Since

$$\gamma_v = \beta_v^{1-1/k} \frac{-p_v}{P_v P_{v-1}} \sum_{r=1}^{v} P_{r-1} a_r,$$

then we have

$$\begin{aligned}
\tau_n &= \sum_{v=1}^{n-1} (\sum_{r=1}^{v} p_{r-1} a_r) \Delta_v (v P_{v-1}^{-1} q_{n-v} \epsilon_v) + (\sum_{r=1}^{n} p_{r-1} a_r) n P_{n-1}^{-1} q_0 \epsilon_n \\
&= \sum_{v=1}^{n-1} (-v \frac{P_v}{p_v} \Delta_v q_{n-v} \epsilon_v \beta_v^{1/k-1} \gamma_v - v q_{n-v-1} \varepsilon_v \beta_v^{1/k-1} \gamma_v \\
&\quad + \frac{P_{v-1}}{p_v} q_{n-v-1} \epsilon_v \beta_v^{1/k-1} \gamma_v \\
&\quad + (v+1) \frac{P_{v-1}}{p_v} q_{n-v-1} \Delta \epsilon_v \beta_v^{1/k-1} \gamma_v) - n \frac{P_n}{p_n} q_0 \varepsilon_n \beta_n^{1/k-1} \gamma_n \\
&= \tau_{n,1} + \tau_{n,2} + \tau_{n,3} + \tau_{n,4} + \tau_{n,5}, \text{ say.}
\end{aligned}$$

In order to prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\tau_{n,r}|^k < \infty, \quad r = 1, 2, 3, 4, 5.$$

Applying Hölder's inequality,

$$\sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k \tau_{n,1}^k$$
$$= \sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\sum_{v=1}^{n-1} -v \frac{P_v}{p_v} \Delta_v q_{n-v} \epsilon_v \beta_v^{1/k-1} \gamma_v|^k$$
$$\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k \sum_{v=1}^{n-1} v^k (\frac{P_v}{p_v})^k |\Delta_v q_{n-v}| |\epsilon_v|^k \beta_v^{1-k} |\gamma_v|^k$$
$$\times \{\sum_{v=1}^{n-1} |\Delta_v q_{n-v}|\}^{k-1}$$
$$\leq 0(1) \sum_{v=1}^{m} v^k (\frac{P_v}{p_v})^k |\epsilon_v|^k \beta_v^{1-k} |\gamma_v|^k \sum_{n=v+1}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\Delta_v q_{n-v}|$$
$$\leq 0(1) \sum_{v=1}^{m} v^k (\frac{\alpha_v}{\beta_v})^{k-1} (\frac{P_v}{p_v})^k (\frac{q_v}{Q_v Q_{v-1}})^k |\varepsilon_v|^k |\gamma_v|^k \sum_{n=v+1}^{m+1} |\Delta_v q_{n-v}|$$
$$\leq 0(1) \sum_{v=1}^{m} v^k (\frac{\alpha_v}{\beta_v})^{k-1} (\frac{P_v}{p_v})^k (\frac{q_v}{Q_v Q_{v-1}})^k |\epsilon_v|^k |\gamma_v|^k.$$

$$\sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\tau_{n,2}|^k$$
$$= \sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\sum_{v=1}^{n-1} -v q_{n-v-1} \epsilon_v \beta_v^{1/k-1} \gamma_v|^k$$
$$\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n})^k Q_{n-1}^{-1} \sum_{v=1}^{n-1} v^k q_{n-v-1} |\epsilon_v|^k \beta_v^{1-k} |\gamma_v|^k \{\sum_{v=1}^{n-1} \frac{q_{n-v-1}}{Q_{n-1}}\}^{k-1}$$
$$\leq 0(1) \sum_{v=1}^{m} v^k |\epsilon_v|^k \beta_v^{1-k} |\gamma_v|^k \sum_{n=v+1}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n})^k \frac{q_{n-v-1}}{Q_{n-1}}$$
$$\leq 0(1) \sum_{v=1}^{m} v^k (\frac{\alpha_v}{\beta_v})^{k-1} (\frac{P_v}{p_v})^k (\frac{q_v}{Q_v Q_{v-1}})^k |\epsilon_v|^k |\gamma_v|^k.$$

$$\sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\tau_{n,3}|^k$$
$$= \sum_{n=2}^{m+1} \alpha_n^{k-1} (\frac{q_n}{Q_n Q_{n-1}})^k |\sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} q_{n-v-1} \epsilon_v \beta_v^{1/k-1} \gamma_v|^k$$

$$\leq \sum_{n=2}^{m+1} \alpha_n^{k-1}(\frac{q_n}{Q_n})^k Q_{n-1}^{-1} \sum_{v=1}^{n-1}(\frac{P_{v-1}}{p_v})^k q_{n-v-1}|\epsilon_v|^k \beta_v^{1-k}|\gamma_v|^k \{\sum_{v=1}^{n-1}\frac{q_{n-v-1}}{Q_{n-1}}\}^{k-1}$$
$$\leq 0(1)\sum_{v=1}^{m}(\frac{P_{v-1}}{p_v})^k|\epsilon_v|^k\beta_v^{1-k}|\gamma_v|^k \sum_{n=v+1}^{m+1}\alpha_n^{k-1}(\frac{q_n}{Q_n})^k\frac{q_{n-v-1}}{Q_{n-1}}$$
$$\leq 0(1)\sum_{v=1}^{m}(\frac{\alpha_v}{\beta_v})^{k-1}(\frac{P_{v-1}}{p_v})^k(\frac{q_v}{Q_vQ_{v-1}})^k|\epsilon_v|^k|\gamma_v|^k(\frac{P_v}{p_v})^k$$

$$\sum_{n=2}^{m+1}\alpha_n^{k-1}(\frac{q_n}{Q_nQ_{n-1}})^k|\tau_{n,4}|^k$$
$$= \sum_{n=2}^{m+1}\alpha_n^{k-1}(\frac{q_n}{Q_nQ_{n-1}})^k|\sum_{v=1}^{n-1}(v+1)\frac{P_{v-1}}{p_v}q_{n-v-1}\Delta\epsilon_v\beta_v^{1/k-1}\gamma_v|^k$$
$$\leq \sum_{n=2}^{m+1}\alpha_n^{k-1}(\frac{q_n}{Q_n})^k Q_{n-1}^{-1}\sum_{v=1}^{n-1}(v+1)^k(\frac{P_{v-1}}{p_v})^k q_{n-v-1}|\Delta\epsilon_v|^k\beta_v^{1-k}|\gamma_v|^k$$
$$\times\{\sum_{v=1}^{n-1}\frac{q_{n-v-1}}{Q_{n-1}}\}^{k-1}$$
$$\leq 0(1)\sum_{v=1}^{m}v^k(\frac{P_{v-1}}{p_v})^k|\Delta\epsilon_v|^k\beta_v^{1-k}|\gamma_v|^k\sum_{n=v+1}^{m+1}\alpha_n^{k-1}(\frac{q_n}{Q_n})^k\frac{q_{n-v-1}}{Q_{n-1}}$$
$$\leq 0(1)\sum_{v=1}^{m}v^k(\frac{\alpha_v}{\beta_v})^{k-1}(\frac{P_v}{p_v})^k(\frac{P_{v-1}}{p_v})^k(\frac{q_v}{Q_vQ_{v-1}})^k|\epsilon_v|^k|\gamma_v|^k$$

$$\sum_{n=1}^{m}\alpha_n^{k-1}(\frac{q_n}{Q_nQ_{n-1}})^k|\tau_{n,5}|^k$$
$$= \sum_{n=1}^{m}\alpha_n^{k-1}(\frac{q_n}{Q_nQ_{n-1}})^k|-n\frac{P_n}{p_n}q_0\varepsilon_n\beta_n\epsilon_n\beta_n^{1/k-1}\gamma_n|^k$$
$$\leq 0(1)\sum_{n=1}^{m}n^k(\frac{\alpha_n}{\beta_n})^{k-1}(\frac{P_n}{p_n})^k(\frac{q_n}{Q_nQ_{n-1}})^k|\epsilon_n|^k|\gamma_n|^k$$

This completes the proof of the theorem.

4. Applications

It is clear that: $|\bar{N},p_n,n|_k = |R,p_n|_k$, $|\bar{N},p_n,P_n/p_n|_k = |\bar{N},p_n|_k$, $|\bar{N},p_n,1|_1 = |\bar{N},p_n|_1$, $|\bar{N},1,n|_k = |C,1|_k$ and $|N,p_n,(\frac{P_{n-1}}{n^{1/k}p_n})^{k/(k-1)}| = |N,p_n|_k$. Also we define $|\bar{R},p_n|_k = |N,p_n,n|_k$.

Theorem D. *Let $q \in M$ such that $\{\alpha_n^{1-1/k} q_n / Q_n Q_{n-1}\}$ nonincreasing. Let*

$$P_n = 0(np_n) \tag{6}$$

and

$$\sum_{n=v}^{\infty} \alpha_n^{k-1} (\frac{q_n}{Q_n})^k \frac{q_{n-v-1}}{Q_{n-1}} = 0\{\alpha_v^{k-1} (\frac{P_v}{p_v}^k (\frac{q_v}{Q_v Q_{v-1}})^k\}. \tag{7}$$

Then a necessary and sufficient condition that $\sum a_n$ is summable $|N, q_n, \alpha_n|_k$ whenever it is summable $|\bar{N}, p_n, \beta_n|_k$, $k \geq 1$, is

$$nP_n q_n = 0\{p_n Q_n Q_{n-1} (\frac{\beta_n}{\alpha_n})^{1-1/k}\}. \tag{8}$$

Proof. Sufficiency. Follows from theorem C by putting $\epsilon_n = 1$.

Necessity. On the lines of Bor [2], put $\frac{q_n}{Q_n Q_{n-1}} T_n = a_n$, $\Delta t_{n-1} = b_n$, then $\frac{\alpha_n^{1-1/k} q_n}{Q_n Q_{n-1}} T_{n,5} = (\frac{\alpha_n}{\beta_n})^{1-1/k} \frac{nP_n q_n q_0}{p_n Q_n Q_{n-1}} \beta_n^{1-1/k} b_n$. It is possible to write the matrix transforming $(\beta_n^{1-1/k} b_n)$ into $(\alpha_n^{1-1/k} a_n)$. Since $|\bar{N}, p_n, \beta_n|_k$ implies $|N, q_n, \alpha_n|_k$, then the matrix $\epsilon(\ell^k; \ell^k)$. By lemma 1, a necessary conditions for this implication is that the elements (in particular the diagonal elements) of this matrix should be bounded. Hence (8).

Corollary 1. *Let $q \in M$. If*

$$P_n = 0(np_n) \tag{9}$$

and

$$\sum_{n=v}^{\infty} \frac{Q_{n-1}^{k-1} q_{n-v-1}}{nQ_n^k} = 0\{v^{-1} (\frac{p_v}{p_v Q_v})^k\}, \tag{10}$$

then a necessary and sufficient condition that

$$|\bar{N}, p_n|_k \Rightarrow |N, q_n|_k,$$

$k \geq 1$, is

$$n/Q_n = 0\{(np_n/P_n)^{1/k}\}.$$

Proof. Follows from theorem D by putting $\alpha_n^{k-1} = (\frac{Q_{n-1}}{n^{1/k} q_n})^k, \beta_n = P_n/p_n$.

Corollary 2. ([1] and [5]). *If*

$$np_n = 0(P_n), \tag{11}$$

then a necessary and sufficient condition that

$$|\bar{N}, p_n|_k \Rightarrow |C, 1|_k,$$

$k \geq 1$, *is (9).*

Proof. With $q_n = 1$, (10) follows from (11). The result follows from corollary 1.

Corollary 3. *Let* $q \in M$. *If (9) holds and*

$$\sum_{n=v}^{\infty} n^{k-1}(\frac{q_n}{Q_n})^k \frac{q_{n-v-1}}{Q_{n-1}} = 0\{v^{k-1}(\frac{P_v}{p_v})^k(\frac{q_v}{Q_v Q_{v-1}})^k\},$$

then a necessary and sufficient condition that

$$|\bar{N}, p_n|_k \Rightarrow |\bar{R}, q_n|_k,$$

$k \geq 1$, *is*

$$n^2 q_n / Q_n Q_{n-1} = 0\{(np_n/P_n)^{1/k}\}.$$

Proof. Follows from theorem D by putting $\alpha_n = n$, $\beta_n = P_n/p_n$.

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