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VANISHING THEOREMS ON JACOBI FORMS OF HIGHER DEGREE

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1. Introduction

For any positive integer $g \in \mathbf{Z}^+$, we let

$$H_g := \{\, Z \in \mathbf{C}^{(g,g)} \,|\, {}^tZ = Z, \quad Im\, Z > 0 \,\}$$

be the Siegel upper half plane of degree g and $\Gamma_g := Sp(g, \mathbf{Z})$ be the Siegel modular group of degree g. Let ρ be an irreducible finite dimensional representation of $GL(g, \mathbf{C})$ and $\mathcal{M}$ be a symmetric half integral positive definite matrix of degree h. It is known ([Z] Theorem 1.8) that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_g is finite dimensional. For the precise definition of $J_{\rho,\mathcal{M}}(\Gamma_g)$, we refer to Definition 2.2. It is a natural question to ask under which conditions the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ vanishes. In this paper, the author gives a vanishing theorem on $J_{\rho,\mathcal{M}}(\Gamma_g)$.

In section 2, we establish the notations and review some properties of Jacobi forms. In section 3, we define the Siegel-Jacobi operator and give the relation between the corank of a Jacobi form and the corank of ρ. In the final section, we establish the Shimura isomorphism and give a vanishing theorem on Jacobi forms using this isomorphism and the vanishing theorem on Siegel modular forms ([W] Satz 2).

Notations:. We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{R})$ and $Z \in H_g$, we set $M < Z > := (AZ + B)(CZ + D)^{-1}$. $\Gamma_g := Sp(g, \mathbf{Z})$ denotes the Siegel modular group of

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degree g. $[\Gamma_g, k]$ (*resp.* $[\Gamma_g, \rho]$) denotes the vector space of all Siegel modular forms of weight k (*resp. of type* ρ). The symbol ":=" means that the expression on the right is the definition of that on the left. We denotes by $\mathbf{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n.

2. Jacobi Forms

In this section, we establish the notations and review some properties of Jacobi forms.

For two positive integers g and h, we consider the ***Heisenberg group***

$$H_{\mathbf{R}}^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \mid \lambda,\, \mu \in \mathbf{R}^{(h,g)},\, \kappa \in \mathbf{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \ symmetric \,\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] \ := \ [(\lambda+\lambda', \mu+\mu'), \kappa+\kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'].$$

We define the semidirect product of $Sp(g,\mathbf{R})$ and $H_{\mathbf{R}}^{(g,h)}$

$$G_{\mathbf{R}}^{(g,h)} := Sp(g,\mathbf{R}) \propto H_{\mathbf{R}}^{(g,h)} \qquad (2.1)$$

endowed with the following multiplication law

$$\begin{aligned}(M,[(\lambda,\mu),\kappa]) &\cdot (M',[(\lambda',\mu'),\kappa']) \\ &:= (MM', [(\tilde{\lambda}+\lambda', \tilde{\mu}+\mu'), \kappa+\kappa' + (\tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda')]),\end{aligned}$$

with $M, M' \in Sp(g,\mathbf{R})$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. The group $G_{\mathbf{R}}^{(g,h)} := Sp(g,\mathbf{R}) \propto H_{\mathbf{R}}^{(g,h)}$ is called the ***Jacobi group***. It is easy to see that $G_{\mathbf{R}}^{(g,h)}$ acts on $H_g \times \mathbf{C}^{(h,g)}$ transitively by

$$(2.3)\quad (M,[(\lambda,\mu),\kappa])\cdot(Z,W) := (M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbf{R}),\ (Z,W) \in H_g \times \mathbf{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g,\mathbf{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbf{R}^{(h,h)}$ be a symmetric half integral matrix of degree h. The *canonical automorphy factor* $I_{\rho,\mathcal{M}}$ for the action of $G_{\mathbf{R}}^{(g,h)}$ on $H_g \times \mathbf{C}^{(h,g)}$ is given by

$$I_{\rho,\mathcal{M}}(\hat{M},(Z,W)) := e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z{}^t\lambda+2\lambda{}^tW+(\kappa+\mu{}^t\lambda)))}\,\rho(CZ+D)^{-1},$$

where $\hat{M} = (M,[(\lambda,\mu),\kappa])$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbf{R})$.
We denote by $C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ the space of all smooth functions with values in V_ρ defined on $H_g \times \mathbf{C}^{(h,g)}$. Then we obtain an action of $G_{\mathbf{R}}^{(g,h)}$ on the space $C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ by putting

$$(2.5)\qquad (f|_{\rho,\mathcal{M}}[\hat{M}])(Z,W) := I_{\rho,\mathcal{M}}(\hat{M},(Z,W))\,f(\hat{M}\cdot(Z,W)),$$

where $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)})$.

A straightforward calculation yields the following.

LEMMA 2.1. *Let* $g_i = (M_i,[(\lambda_i,\mu_i),\kappa_i]) \in G_{\mathbf{R}}^{(g,h)}$ $(i = 1,2)$. *For any* $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$, *we have*

$$(2.6)\qquad (f|_{\rho,\mathcal{M}}[g_1])|_{\rho,\mathcal{M}}[g_2] = f|_{\rho,\mathcal{M}}[g_1 g_2].$$

DEFINITION 2.2. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbf{Z}}^{(g,h)} := \{\,[(\lambda,\mu),\kappa] \in H_{\mathbf{R}}^{(g,h)} \mid \lambda,\mu \in \mathbf{Z}^{(h,g)},\ \kappa \in \mathbf{Z}^{(h,h)}\,\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \propto H_{\mathbf{Z}}^{(g,h)}$.
(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbf{Z}^{(g,h)}} c(T,R)\, e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by Koecher principle(see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = \mathbf{C}$, $\rho(A) = (\det A)^k\ (k \in \mathbf{Z},\ A \in GL(g,\mathbf{C}))$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma)$.

Ziegler([Z] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

From now on, we assume that Γ is a normal subgroup of Γ_g of finite index. If $M \in \Gamma_g$, then $\Gamma^M := M^{-1}\Gamma M$ is a subgroup of Γ_g of finite index. It is easy to show that if $f \in J_{\rho,\mathcal{M}}(\Gamma)$, then $f|_{\rho,\mathcal{M}}[M] \in J_{\rho,\mathcal{M}}(\Gamma^M)$. Thus $f|_{\rho,\mathcal{M}}[M]$ has the Fourier expansion of the form

$$(2.7) \qquad (f|_{\rho,\mathcal{M}}[M])(Z,W) = \sum_{T,R} c_M(T,R)\, e^{\frac{2\pi i}{\lambda}\sigma(TZ)} \cdot e^{2\pi i \sigma(RW)},$$

where T runs over the set of all semipositive half integral matrices of degree g, R runs over the set of $g \times h$ integral matrices and $\lambda = \lambda_{\Gamma^M} \in \mathbf{Z}$ is a suitable integer.

DEFINITION 2.3. Let ρ be an irreducible finite dimensional representation of $GL(g,\mathbf{C})$. Then ρ is determined uniquely by its highest weight $(\lambda_1, \cdots, \lambda_g) \in \mathbf{Z}^g$ with $\lambda_1 \geq \cdots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ. The number of λ_i's such that $\lambda_i = k(\rho) = \lambda_g\ (1 \leq i \leq g)$ is called the *corank* of ρ, denoted by *corank* (ρ).

DEFINITION 2.4. Let $f \in J_{\rho,\mathcal{M}}(\Gamma)$ be a nonvanishing Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ. We define the ***corank*** of f as follows:

$$corank\,(f) := g - min_T\,(rank\,(T)),$$

where T runs over the set of all semipositive half integral matrices of degree g such that $c_M(T,R) \neq 0$ for at least one $M \in \Gamma_g$.

Let $T = (t_{ij})$ be a semipositive symmetric matrix of degree g. We write $r(T) = d$ if $t_{g-d,g-d}$ is the last diagonal element distinct from zero. Since $T \geq 0$, T must be of the form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 \geq 0, \quad T_1 \in \mathbf{R}^{(g-d,g-d)}. \tag{2.8}$$

We note that T_1 is not invertible in general.

LEMMA 2.5. *Let $0 \neq f \in J_{\rho,\mathcal{M}}(\Gamma)$. Then for all T with $c_M(T,R) \neq 0$, we have $r(T) \leq corank\,(f)$. There exists $M \in \Gamma_g$ and T with $c_M(T,R) \neq 0$ such that*

$$r(T) = corank\,(\rho).$$

The proof of the above lemma is obvious.

3. The Siegel-Jacobi Operator

In this section, we define the Siegel-Jacobi operator and give the relation between the corank of a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma)$ and that of ρ using the Siegel-Jacobi operator.

Let $\rho : GL(g,\mathbf{C}) \longrightarrow GL(V_\rho)$ be an irreducible rational representation of $GL(g,\mathbf{C})$ on a finite dimensional complex vector space V_ρ. Let $0 \leq r \leq g-1$. Now for a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ defined on $H_g \times \mathbf{C}^{(h,g)}$, we define $\Psi_{g,r}f \in \mathcal{O}(H_r \times \mathbf{C}^{(h,r)}, V_\rho)$ by

$$(\Psi_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W,0)\right), \tag{3.1}$$

where $Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. We note that the above limit (3.1) always exists because a Jacobi form f admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z,W) \in H_g \times \mathbb{C}^{(h,g)} \,|\, Im\, Z \geq Y_0 > 0,\ W \in K \subset \mathbb{C}^{(h,g)}\ compact\}.$$

The operator $\Psi_{g,r}$ is called the *Siegel-Jacobi operator.*

As before, we assume that Γ is a normal subgroup of Γ_g of finite index. For $M \in \Gamma_g$, $\Gamma^M := M^{-1}\Gamma M$ is a subgroup of Γ_g of finite index. If $f \in J_{\rho,\mathcal{M}}(\Gamma)$, then $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the form (2.7). Let $c_M(T,R)$ be a Fourier coefficient of $f|_{\rho,\mathcal{M}}[M]$ with $r(T) = g - r$. Let $\mathcal{U}$ be the subgroup of $GL(g,\mathbb{Z})$ consisting of $U \in GL(g,\mathbb{Z})$ such that

$$M_U = \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma^M \subset \Gamma_g.$$

Since $(f|_{\rho,\mathcal{M}}[M])|_{\rho,\mathcal{M}}[M_U] = f|_{\rho,\mathcal{M}}[M]$, applying the Fourier expansion (2.7), we have

$$c_M(UT\,{}^tU,\ UR) = \rho(U)\, c_M(T,R). \tag{3.2}$$

For $k = 1, \cdots, g$, we let

$$G_{g,k} := \left\{ \begin{pmatrix} E_{g-k} & * \\ 0 & * \end{pmatrix} \in GL(g,\mathbb{C}) \right\}.$$

Then $G_{g,k} = GL(k,\mathbb{C}) \propto N$, where N is a unipotent radical of the group $G_{g,k}$. Then for any $U \in G_{g,g-r}$, we have

$$UT\,{}^tU = T, \quad UR = R. \tag{3.3}$$

Indeed, T is of the form (2.8). Since $\begin{pmatrix} \frac{1}{\lambda}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$ with $\lambda = \lambda_{\Gamma^M}$, R is of the form

$$R = \begin{pmatrix} R_1 & \\ 0 & \end{pmatrix}, R_1 \in \mathbb{Z}^{(r,h)}.$$

According to (3.3), we obtain

$$c_M(T,R) = \rho(U)\, c_M(T,R), U \in \mathcal{U} \cap G_{g,g-r}. \tag{3.4}$$

We observe that the Zariski closure of $\mathcal{U} \cap G_{g,g-r}$ in $G_{g,g-r}$ contains the subgroup $G(g-r) := SL(g-r, \mathbf{C}) \propto N$. Thus $c_M(T,R)$ is invariant under all $U \in G(g-r)$ in the sense of (3.4). For $k = 1, \cdots, g$, we put

$$V_\rho^{G(k)} := \{, v \in V_\rho, | \rho(g)v = v \text{ for all } g \in G(k), \} \tag{3.5}$$

Here $G(k) := SL(k, \mathbf{C}) \propto N$. Then according to [W], we have

$$V_\rho^{G(k)} \cong \begin{cases} (\lambda_1, \cdots, \lambda_{g-k}) & if corank, (\rho) \geq k, \\ 0 & otherwise. \end{cases} \tag{3.6}$$

That is, if $V_\rho^{G(k)} \neq 0, V_\rho^{G(k)}$ is an irreducible finite dimensional representation of $GL(g-k, \mathbf{C})$.

Let $V_\rho^{(r)}$ be the subspace of V_ρ generated by the values $\{\, \Psi_{g,r} f(Z,W) \mid f \in J_{\rho,\mathcal{M}}(\Gamma),\ (Z,W) \in H_g \times \mathbf{C}^{(h,g)} \}$. If $V_\rho^{(r)} \neq 0$, according to (3.1), (3.4) and (3.6),

$$V_\rho^{G(g-r)} = V_\rho^{(r)}. \tag{3.7}$$

Thus $V_\rho^{(r)}$ is invariant under

$$\left\{ \left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) : g \in GL(r, \mathbf{C}) \right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $GL(r, \mathbf{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(g)v := \rho\left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) v, \quad g \in GL(r, \mathbf{C}), \quad v \in V_\rho^{(r)}. \tag{3.8}$$

So far we have proved

LEMMA 3.1. *Let ρ be irreducible. Then $(\rho^{(r)}, V_\rho^{(r)})$ is an irreducible finite dimensional representation of $GL(r, \mathbb{C})$.*

For all $0 \neq c_M(T, R) \in V_\rho^{G(g-r)}$, we have $r(T) \leq corank(\rho)$ by (3.6). By Lemma 2.5, we have $corank(f) \leq corank(\rho)$.

Thus we have

THEOREM 3.2. *Let $0 \neq f \in J_{\rho,\mathcal{M}}(\Gamma)$ be a nonvanishing Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ. Then we have*

$$corank(f) \leq corank(\rho).$$

For more results on the Siegel-Jacobi operators, we refer to [Y1] or [Y2].

4. Vanishing Theorems

In this section, we establish the Shimura isomorphism and using this isomorphism we prove a vanishing theorem.

Let S be a symmetric, positive definite integral matrix of degree h and let $a, b \in \mathbb{Q}^{(h,g)}$. We consider

$$\vartheta_{S,a,b}(Z, W) := \sum_{\lambda \in Z^{(h,g)}} e^{\pi i \sigma(S((\lambda+a)Z^t(\lambda+a)+2(\lambda+a)^t(W+b)))} \tag{4.1}$$

with characteristic (a, b) converging uniformly on any compact subset of $H_g \times \mathbb{C}^{(h,g)}$.

Let $\mathcal{M}$ be a symmetric, positive definite and half-integral matrix of degree h and let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$. We observe that $\#(\mathcal{N}) = \{det(2\mathcal{M})\}^g$. An easy application of the Poisson summation formula gives

LEMMA 4.1. *For $a \in \mathcal{N}$, we have*

$$\begin{aligned}(4.2)\quad &\vartheta_{2\mathcal{M},a,0}(-Z^{-1}, WZ^{-1}) \\ &= \{det(2\mathcal{M})\}^{-\frac{g}{2}} \left\{det\left(\frac{Z}{i}\right)\right\}^{\frac{h}{2}} e^{2\pi i\sigma(\mathcal{M}WZ^{-1}{}^tW)} \\ &\times \sum_{b\in\mathcal{N}} e^{-2\pi i\sigma(2\mathcal{M}b^ta)}\,\vartheta_{2\mathcal{M},b,0}(Z,W).\end{aligned}$$

Here we denote by

$$h(Z) := \sqrt{det\left(\frac{Z}{i}\right)}$$

the unique holomorphic function on H_g satisfying the following properties

$$\text{(a)}\qquad h(Z)^2 = det\left(\frac{Z}{i}\right)$$

$$\text{(b)}\qquad h(iE_g) = +1$$

and for any integer $r \in \mathbb{Z}$, we put

$$\left\{det\left(\frac{Z}{i}\right)\right\}^{\frac{r}{2}} := h(Z)^r = \left\{\sqrt{det\left(\frac{Z}{i}\right)}\right\}^r.$$

COROLLARY 4.2. *Let $2\mathcal{M}$ be unimodular. Then $\vartheta_{2\mathcal{M},0,0}(Z,W)$ is a Jacobi form of weight $\frac{h}{2}$ and index $\mathcal{M}$.*

We fix an element $Z_0 \in H_g$. We denote by $T_{\mathcal{M}}(Z_0)$ the vector space of all holomorphic functions $\varphi : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ satisfying the condition

$$(4.3)\qquad \varphi(W + \lambda Z_0 + \mu) = e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_0{}^t\lambda + 2\lambda{}^tW))}\varphi(W)$$

for every $\lambda, \mu \in \mathbb{Z}^{(h,g)}$. Then it is easy to show that the functions

$$(4.4)\qquad \{\,\vartheta_{2\mathcal{M},a,0}(Z_0, W)\,|\,a \in \mathcal{N}\,\}$$

form a basis of $T_{\mathcal{M}}(Z_0)$ and its dimension is clearly $\{det\,(2\mathcal{M})\}^g$ (cf. J. Igusa [I]). Let $\rho : GL(g, \mathbf{C}) \longrightarrow GL(V_\rho)$ be an irreducible rational representation of $GL(g, \mathbf{C})$ on a finite dimensional complex vector space V_ρ. If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma)$, it is easy to see that each component of $\phi(W) := f(Z_0, W)$ satisfies the relation (4.3). So we may write

$$(4.5)\quad f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W),\quad Z\in H_g,\quad W\in\mathbf{C}^{(h,g)},$$

where $\mathcal{N}$ is a complete system of representatives of the cosets $(2\mathcal{M})^{-1}$ $\mathbf{Z}^{(h,g)}/\mathbf{Z}^{(h,g)}$ and $\{f_a : H_g \longrightarrow V_\rho \,|\, a \in \mathcal{N}\}$ are uniquely determined holomorphic functions on H_g.

LEMMA 4.3. *Each $f_a(Z)\,(a \in \mathcal{N})$ is holomorphic.*

Proof. Since f and $\vartheta_{2\mathcal{M},a,0}(Z,W)\,(a\in\mathcal{N})$ are holomorphic,

$$\sum_{a\in\mathcal{N}} \frac{\partial f_a(Z)}{\partial \bar{Z}_{ij}}\cdot\vartheta_{2\mathcal{M},a,0}(Z,W) = 0,\quad Z = (Z_{ij}) = {}^tZ.$$

Since $\vartheta_{2\mathcal{M},a,0}(Z,W)(a\in\mathcal{N})$ form a basis of $T_{\mathcal{M}}(Z)$ as functions on $\mathbf{C}^{(h,g)}$, $\frac{\partial f_a(Z)}{\partial \bar{Z}_{ij}} = 0$ for all $Z\in H_g$ and $a\in\mathcal{N}$. Hence each $f_a(Z)\,(a\in\mathcal{N})$ is holomorphic.

According to Lemma 4.1, we have

(4.6)

$$f_a(-Z^{-1}) = \left\{det,\left(\frac{Z}{i}\right)\right\}^{-\frac{h}{2}}\cdot\{\rho(-Z)\}\cdot\{det,(2\mathcal{M})\}^{-\frac{g}{2}} \times \sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}a^tb)}\cdot f_b(Z)$$

and

$$(4.7)\qquad f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}aS^ta)}\cdot f_a(Z), S = {}^tS\in\mathbf{Z}^{(g,g)}.$$

We note that the Fourier coefficients $c(T,R)$ of $\vartheta_{2\mathcal{M},a,0}(Z,W)$ are given by

$$c(T,R)=\begin{cases} 1 & if\ {}^{\exists}\lambda\in \mathbf{Z}^{(h,g)} s.t.\\ & {}^{t}(\lambda+a,E_h)\mathcal{M}(\lambda+a,E_h)=\begin{pmatrix} T & \frac{1}{2}R\\ \frac{1}{2}\,{}^{t}R & \mathcal{M}\end{pmatrix}\\ 0 & otherwise\end{cases}$$

We observe that $c(T,R)\neq 0$ implies $4T-R\mathcal{M}^{-1}\,{}^{t}R=0$. By an easy argument, we see that the functions $\{f_a|a\in\mathcal{N}\}$ must have the Fourier expansions of the form

$$(4.8)\qquad f_a(Z)=\sum_{\substack{T={}^{t}T\geq 0\\ half\ integral}} c(T)\cdot e^{2\pi i\sigma(TZ)}$$

Conversely, suppose there is given a family $\{f_a|a\in\mathcal{N}\}$ of holomorphic functions $f_a : H_g \longrightarrow V_\rho$ satisfying the transformation laws (4.6), (4.7) and the cusp condition (4.8). Then we obtain a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ by defining $f(Z,W)$ via the equation (4.5). So far we have proved the Shimura isomorphism:

THEOREM 1 (SHIMURA). *The equation (4.5) gives an isomorphism between $J_{\rho,\mathcal{M}}(\Gamma_g)$ and the vector space of V_ρ-valued Siegel modular forms of half integral weight satisfying the transformation laws (4.6), (4.7) and the cusp condition (4.8).*

REMARK 4.4. Theorem 1 may be also formulated for Jacobi forms on a subgroup $\Gamma\subset\Gamma_g$ of finite index.

COROLLARY 4.5.. *If $2k<rank(\mathcal{M})$, then we have $J_{k,\mathcal{M}}(\Gamma)=0$.*

Proof. The proof follows from the fact that the irreducible representation $(det)^{k-\frac{1}{2}rank(\mathcal{M})}$ of $GL(g,\mathbf{C})$ is not a polynomial representation. *q.e.d.*

COROLLARY 4.6. *Let $2\mathcal{M}$ be unimodular and $k \cdot g$ be odd. Then* $J_{k,\mathcal{M}}(\Gamma_g) = 0$.

Proof. It follows immediately from (4.6) and the fact that $h \equiv 0 \,(mod\, 8)$. *q.e.d.*

COROLLARY 4.7. *Let $2\mathcal{M}$ be unimodular. We assume that ρ satisfies the following condition (4.9):*

$$\rho(A) = \rho(-A) \quad \text{for all } A \in GL(g, \mathbf{C}). \tag{4.9}$$

Then we have

$$J_{\rho,\mathcal{M}}(\Gamma) = [\Gamma, \tilde{\rho}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W) \cong [\Gamma, \tilde{\rho}], \tag{4.10}$$

where $\tilde{\rho} = \rho \otimes det^{-\frac{h}{2}}$. In particular, if $k \cdot g$ is even,

$$J_{k,\mathcal{M}}(\Gamma) = [\Gamma, k - \frac{h}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W) \cong [\Gamma, k - \frac{h}{2}]. \tag{4.11}$$

Proof. The proof of (4.10) follows from (4.6), (4.7) and (4.8). The representation $det^k : GL(g, \mathbf{C}) \longrightarrow \mathbf{C}^\times$ defined by $det^k(A) = (det\,(A))^k$ satisfies the condition (4.9). Hence (4.11) follows from (4.12). *q.e.d.*

EXAMPLE 4.8. We give several examples of the irreducible representations which satisfies the condition (4.9).

(a) If $k \cdot g$ is even, then the polynomial representation $\rho : GL(g, \mathbf{C}) \longrightarrow \mathbf{C}^\times$ defined by $\rho(A) := (det\, A)^k \,(A \in GL(g, \mathbf{C}))$ satisfies the condition (4.9).

(b) The polynomial representation ρ of $GL(g, \mathbf{C})$ on the symmetric product $Symm^2(\mathbf{C}^g)$ of $\mathbf{C}^g$ defined by

$$\rho(A)Z := AZ\,{}^tA, \quad A \in GL(g, \mathbf{C}), \quad Z \in Symm^2(\mathbf{C}^g)$$

satisfies the condition (4.9). It is obvious that ρ is irreducible. This representation is important geometrically because it is related with holomorphic 1-forms on H_g invariant under Γ_g.

(c) The polynomial representation ρ of $GL(g, \mathbf{C})$ on $Symm^2(\mathbf{C}^g)$ defined by

$$\rho(A)Z := (det\, A)^{g+1} A^{-1} Z\, {}^tA^{-1}, \quad A \in GL(g, \mathbf{C}), \quad Z \in Symm^2(\mathbf{C}^g)$$

satisfies the condition (4.9). It is easy to see that ρ is irreducible. This representation is also important geometrically because it is connected with holomorphic $(N-1)$-forms on H_g invariant under Γ_g, where $N = \frac{g(g+1)}{2} - 1$.

Now we prove a vanishing theorem on Jacobi forms.

THEOREM 2. *Let $2\mathcal{M}$ be an even unimodular positive definite matrix of degree h. Let $\rho = (\lambda_1, \cdots, \lambda_g)$ be an irreducible finite dimensional representation of $GL(g, \mathbf{C})$. Let $\lambda(\rho)$ be the number of λ_i's such that $\lambda_i = k(\rho) + 1 = \lambda_g + 1$, $1 \leq i \leq g$. Assume that ρ satisfies the following conditions:*

(a) ρ satisfies the condition (4.9);

(b) $\lambda(\rho) < 2\,(g - k(\rho) - corank\,(\rho)) + rank\,(\mathcal{M})$.

Then $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$.

Proof. It is easily seen that $corank\,(\rho \otimes det^{-\frac{h}{2}}) = corank\,(\rho)$ and $\lambda(\rho \otimes det^{-\frac{h}{2}}) = \lambda(\rho)$. According to [W] Satz 2, we have $[\Gamma_g, \rho \otimes det^{-\frac{h}{2}}] = 0$. By corollary 4.7, we have $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$. *q.e.d.*

COROLLARY 4.9. *Let $2\mathcal{M}$ be as above in Theorem 2. Assume that $2k(\rho) \leq g + rank\,(\mathcal{M}) - 2\,corank\,(\rho)$. Then $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$.*

Proof. It follows immediately from Theorem 2 and the fact that $\lambda(\rho)$ is less than g. *q.e.d.*

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