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A CLASS OF CONDITIONAL WIENER INTEGRALS

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1. Introduction

Let $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ denote Wiener space where $C_0[0,T]$ is the space of all continuous functions x on $[0,T]$ with $x(0) = 0$. Many physical problem can be formulated in terms of the conditional Wiener integral $E[F|X]$ of the functional defined on $C_0[0,T]$ of the form

$$F(x) = \exp\{-\int_0^T V(x(t))dt\} \tag{1.1}$$

where $X(x) = x(T)$ and V is a sufficiently smooth function on $\mathbf{R}$. Indeed, it is known [see [3],[4],[7]] that the function U defined on $\mathbf{R} \times [0,T] \times \mathbf{R}$ by

$$U(\xi, t; \xi_0) = \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(\xi - \xi_0)^2}{2t}\}E[F(x(\cdot) + \xi_0)|x(t) = \xi - \xi_0]$$

is the Green's function for the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial \xi^2}U - VU.$$

So it is of interest to obtain formulas for evaluating such conditional Wiener integrals.

J.Yeh [8] derived several Fourier inversion formulas for conditional Wiener integrals and then used the formulas to evaluate conditional Wiener integrals. Recently, Park and Skoug ([5],[6]) obtained a simple formula of another type for evaluating conditional Wiener and Yeh-Wiener integrals. Chung and Kang [2] defined abstract Wiener space

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version of conditional Wiener integrals and then obtained evaluation formulas for conditional abstract Wiener integral of various functions which include some results given in [5],[6].

In this paper, we consider a class of functions V of the form $V(s,\xi) = -\frac{\alpha}{2}\xi^2 + \alpha\beta q(s)\xi$, where $q \in L^2[0,T]$ and α, β are complex, and give explicit formulas of the conditional Wiener integral of the functions F of the form (1.1) for the class of V's.

2. Preliminaries

For the partition $\tau = \tau_n = \{t_1, ..., t_n\}$ of [0,T] with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_\tau : C_0[0,T] \to \mathbf{R}^n$ be defined by $X_\tau(x) = (x(t_1), \cdots, x(t_n))$. Let $\mathcal{B}(\mathbf{R}^n)$ be the σ-algebra of Borel sets in $\mathbf{R}^n$. Then a set of the type

$$I = \{x \in C_0[0,T] : X_\tau(x) \in B\} \equiv X_\tau^{-1}(B), \quad B \in \mathcal{B}(R^n)$$

is called a Borel cylinder set. The collection $\mathcal{F}$ of such a set forms an algebra of subsets of $C_0[0,T]$. It is well known that the set function m_w on $\mathcal{F}$ defined by

$$m_w(I) = \int_B K(\tau, \vec{\xi})d\vec{\xi},$$

where

$$K(\tau,\vec{\xi}) = \left\{\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right\}^{-1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{t_j - t_{j-1}}\right\},$$

with $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$ and $\xi_0 = 0$, is a probability measure and thus m_w is extended to the Borel σ-algebra $\mathcal{B}(C_0[0,T])$ generated by $\mathcal{F}$.

Let F be a complex-valued ($\mathbf{C}$-valued) integrable function on $C_0[0,T]$. Let $\mathcal{F}(X_\tau)$ be the σ-algebra generated by the set $\{X_\tau^{-1}(B) : B \in \mathcal{B}(R^n)\}$. Then, by the definition of conditional expectation, the conditional expectation of F given by $\mathcal{F}_\tau$, written $E[F|X_\tau]$, is any real valued $\mathcal{F}_\tau$-measurable function on $C_0[0,T]$ such that

$$\int_E F dm_w = \int_E E[F|X_\tau] dm_w \quad \text{for} \quad E \in \mathcal{F}_\tau.$$

It is well known that there exists a Borel measurable and P_{X_τ}-integrable function Ψ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), P_{X_\tau})$ such that $E[F|X_\tau] = \Psi \circ X_\tau$ and P_{X_τ} is the probability distribution of X_τ defined by $P_{X_\tau}(A) = m_w(X_\tau^{-1}(A))$ for $A \in \mathcal{B}(\mathbf{R}^n)$. Following Yeh [8], the function $\Psi(\vec{\xi})$, written $E[F|X_\tau = \vec{\xi}]$, is called the conditional Wiener integral of F given X_τ.

For a given partition $\tau = \tau_n$ of $[0,T]$ and $x \in C_0[0,T]$, define the polygonal function $[x]$ on $[0,T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})),$$

for $t \in [t_{j-1}, t_j], j = 1, \cdots, n$. Likewise, for each $\vec{\xi} = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$, define the polygonal function $[\vec{\xi}]$ of $\vec{\xi}$ on $[0,T]$ by

$$[\vec{\xi}](t) = \xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}),$$

for $t \in [t_{j-1}, t_j]$, $j = 1, \cdots, n$, and $\xi_0 = 0$.

The following theorem, due to Park and Skoug [5], is a evaluation formular for conditional Wiener integrals.

THEOREM 2.1. *Let F be an integrable function on $C_0[0,T]$. Then for $\vec{\xi} \in \mathbf{R}^n$,*

$$E[F(x)|X_\tau(x) = \vec{\xi}] = \int_{C_0[0,T]} F(x - [x] + [\vec{\xi}])dm_w(x).$$

We note that a real valued function Y on $[0,T] \times C_0[0,T]$ defined by

$$Y(t,x) \equiv y(t) = x(t) - \frac{t}{T}x(T)$$

is a pinned Wiener process on $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ and $[0,T]$ with $y(0) = 0$ and $y(T) = 0$. This process $\{y(t), 0 \le t \le T\}$ induces the Gaussian measure, called the pinned Wiener measure m_p, on $C_0^0[0,T] = \{x \in C_0[0,T] | x(T) = 0\}$, which is uniquely determined by mean function $E[y(t)] = 0$ for every t $\in$ [0,T] and covariance function $E[y(s), y(t)] = \min\{s,t\} - \dfrac{st}{T}$.

In the following theorem, we gives a convenient formula for evaluating conditional Wiener integrals of the function involving quadratic functional.

THEOREM 2.2. *Let F be an integrable function on $C_0[0,T]$. Then for $0 < t_1 < T$ and $\xi, \xi_1 \in \mathbf{R}$,*

$$E[F(x)|x(t_1) = \xi_1, x(T) = \xi] = \int_{C_0^0[0,T-t_1]} F(y+g)dm_p(y),$$

where $g(t) = \frac{t}{T-t_1}(\xi - \xi_1) + \xi_1, t \in [0, T-t_1]$.

In particular if $t_1 = 0$, then

$$E[F(x)|x(T) = \xi] = \int_{C_0^0[0,T]} F(y+h)dm_p(y),$$

where $h(t) = \frac{t}{T}\xi$, $t \in [0,T]$.

Proof. The proof easily follows from the fact that

$$E[F(\cdot)|x(t_1) = \xi_1, x(T) = \xi] = E[F(x(\cdot) + \xi_1)|x(T-t_1) = \xi - \xi_1],$$

Theorem 2.1, and the change of variable formula.

3. Main Theorem

Let k be the covariance function of the pinned Wiener process $\{y(t) : t \in [0,T]\}$, that is, k is the function on $[0,T] \times [0,T]$ defined by

$$k(s,t) = \min\{s,t\} - \frac{st}{T}. \tag{3.1}$$

Let A be the integral operator on $L^2[0,T]$ (the space of real valued square integrable function on [0,T]) defined by

$$Af(s) = \int_0^T k(s,t)f(t)dt, \quad s \in [0,T], \quad f \in L^2[0,T]. \tag{3.2}$$

Then it can be shown that the orthonormal eigen-functions $\{e_n\}$ of A are given by

$$e_n(s) = \sqrt{\frac{T}{2}}\sin(\frac{n\pi}{T}s) \tag{3.3}$$

and the corresponding eigen-value $\{\alpha_n\}$ are given by

$$\alpha_n = \frac{T^2}{n^2\pi^2}. \tag{3.4}$$

Further, it can be shown that $\{e_n\}$ is a basis of $L^2[0,T]$, and that A is a trace class operator on $L^2[0,T]$. The Karhunen - Loeve theorem [1] shows that the Fourier series representation of the pinned Wiener process $\{y(t) : t \in [0,T]\}$ is given by

$$y(t) = \sum_{n=1}^{\infty} z_n e_n(t), \qquad 0 \le t \le T \tag{3.5}$$

where z_n's are orthogonal Gaussian random variables with $E[z_n] = 0$ and
$E[z_n^2] = \alpha_n$.

LEMMA 3.1. *For* $\alpha > 0, t \in [0,T]$,

$$\sum_{n=1}^{\infty} \frac{T}{n^2\pi^2 + \alpha T^2} \cos(\frac{n\pi}{T} t) = \frac{\cosh\sqrt{\alpha}(T-t)}{2\sqrt{\alpha}\sinh\sqrt{\alpha}t} - \frac{1}{2\alpha T}. \tag{3.6}$$

Proof. To proof this lemma, we use a known result that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi \cos(ax)}{2a\sin(a\pi)}, \quad -\pi \le x \le \pi,$$

where a is not an integer. If we let $a = i\sqrt{\alpha}T/\pi$ and $x = \pi(T-t)/T$, then

$$\sum_{n=1}^{\infty} (-1)^n \frac{\pi^2}{n^2\pi^2 + \alpha T^2} \cos(n\pi - \frac{n\pi}{T} t) = \frac{\pi^2 \cosh(\sqrt{\alpha}(T-t))}{2\sqrt{\alpha}T\sinh\sqrt{\alpha}T} - \frac{\pi^2}{2\alpha T^2}.$$

Hence we obtain

$$\sum_{n=1}^{\infty} \frac{T}{n^2\pi^2 + \alpha T^2} \cos(\frac{n\pi}{T} t) = \frac{\cosh(\sqrt{\alpha}(T-t))}{2\sqrt{\alpha}\sinh\sqrt{\alpha}T} - \frac{1}{2\alpha T}.$$

LEMMA 3.2. *For $\alpha > 0$, let*

$$R(s,t,\alpha) = \sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} e_n(s)e_n(t), \quad s,t \in [0,T]$$

where α_n and e_n are as in (3.3) and (3.4). Then

$$R(s,t,\alpha) = \begin{cases} \dfrac{\sinh\sqrt{\alpha}(T-t)\sinh\sqrt{\alpha}s}{\sqrt{\alpha}\sinh\sqrt{\alpha}T}, & s \le t; \\ \dfrac{\sinh\sqrt{\alpha}(T-s)\sinh\sqrt{\alpha}t}{\sqrt{\alpha}\sinh\sqrt{\alpha}T}, & s \ge t. \end{cases} \tag{3.7}$$

Proof. Using (3.3), (3.4) and Lemma 3.1 we have

$$\begin{aligned} R(s,t,\alpha) &= \frac{T^2}{n^2\pi^2+\alpha T^2}\frac{2}{T}\sin(\frac{n\pi}{T}s)\sin(\frac{n\pi}{T}t) \\ &= \frac{T}{n^2\pi^2+\alpha T^2}[\cos[\frac{n\pi}{T}(s-t)] - \cos[\frac{n\pi}{T}(s+t)]] \\ &= \frac{1}{2\sqrt{\alpha}\sinh\sqrt{\alpha}T}[\cosh\sqrt{\alpha}(T-|s-t|)) - \cosh\sqrt{\alpha}(T-|s+t|)] \\ &= \begin{cases} \dfrac{\sinh\sqrt{\alpha}(T-t)\sinh\sqrt{\alpha}s}{\sqrt{\alpha}\sinh\sqrt{\alpha}T}, & s \le t; \\ \dfrac{\sinh\sqrt{\alpha}(T-s)\sinh\sqrt{\alpha}t}{\sqrt{\alpha}\sinh\sqrt{\alpha}T}, & s \ge t. \end{cases} \end{aligned}$$

THEOREM 3.3. *Let F be a measurable function on $C_0[0,T]$ defined by*

$$F(x) = \exp\{-\frac{1}{2}\alpha\int_0^T x^2(s)ds + \alpha\beta\int_0^T q(s)x(s)ds\}, x \in C_0[0,T],$$

where $Re\alpha > -\frac{\pi^2}{2T^2}$, $\beta \in \mathbf{C}$, *and* $q \in L^2[0,T]$. *Then for* $\xi, \xi_1 \in \mathbf{R}$,

$$E\big[\exp\{-\frac{\alpha}{2}\int_{t_1}^{T} x^2(s)ds + \alpha\beta\int_{t_1}^{T} q(s)x(s)ds\}\,\big|x(t_1) = \xi_1, x(T) = \xi\big]$$
$$= \Big(\frac{\sqrt{\alpha}(T-t_1)}{\sinh\sqrt{\alpha}(T-t_1)}\Big)^{\frac{1}{2}} \cdot \exp\{\frac{(\xi-\xi_1)^2}{2(T-t_1)}\}$$
$$\cdot \exp\{-\frac{\sqrt{\alpha}}{2}\coth\sqrt{\alpha}(T-t_1)(\xi^2+{\xi_1}^2) + \frac{\sqrt{\alpha}\xi\xi_1}{\sinh\sqrt{\alpha}(T-t_1)}\}$$
$$\cdot \exp\{\alpha\beta(\xi-\xi_1)\int_0^{T-t_1}\Big(\frac{\sinh\sqrt{\alpha}t}{\sinh\sqrt{\alpha}(T-t_1)} + \frac{\xi_1}{\xi-\xi_1}\Big)q(t+t_1)dt\}$$
$$\cdot \exp\{\frac{\alpha^2\beta^2}{2}\int_0^{T-t_1}\int_0^{T-t_1} R(s,t,\alpha)q(s+t_1)q(t+t_1)dsdt\}.$$

Proof. We first note that for Re$\alpha > -\frac{\pi^2}{2T^2}$, $\exp\{-\frac{1}{2}\alpha\int_0^T x^2(s)ds\}$ is square Wiener integrable, and that for any $z \in \mathbf{C}$, $\exp\{z\int_0^T q(s)x(s)ds\}$ is square Wiener integrable. Hence F is Wiener integrable for Re$\alpha > -\frac{\pi^2}{2T^2}$ and any $\beta \in \mathbf{C}$. So F is conditional Wiener integrable for the given $x(t_1) = \xi_1$ and $x(T) = \xi$. By Theorem 2.2, we have, for $\xi, \xi_1 \in \mathbf{R}$

$$E\big[\exp\{-\frac{\alpha}{2}\int_{t_1}^{T} x^2(s)ds + \alpha\beta\int_{t_1}^{T} q(s)x(s)ds\}\,\big|x(t_1) = \xi_1, x(T) = \xi\big]$$
$$= \int_{C_0^0[0,T-t_1]} \exp\{-\frac{\alpha}{2}\int_0^{T-t_1}(y(s)+g(s))^2 ds$$
$$+ \alpha\beta\int_0^{T-t_1} q(s+t_1)(y(s)+g(s))ds\}dm_p(y)$$

where $g(t) = \frac{\xi-\xi_1}{T-t_1}t + \xi_1, t \in [0, T-t_1]$. Hence the proceeding equals

$$(3.8)\quad \int_{C_0^0[0,T-t_1]} \exp\{-\frac{1}{2}\alpha\sum_{n=1}^{\infty}\big[(z_n+g_n)^2 - 2\beta(q_n(z_n+g_n))\big]\}dm_p(y)$$

where $y(t)=\sum_{n=1}^{\infty} z_n e_n(t)$ is the Fourier series representation of function y in $C_0^0[0,T-t_1]$ as in (3.5), $g(t)=\sum_{n=1}^{\infty} g_n e_n(t)$, and $q(t+t_1)=\sum_{n=1}^{\infty} q_n e_n(t)$. Since z_n's are independent Gaussian random variables with mean 0 and variance α_n, (3.8) equals

$$\prod_{n=1}^{\infty}\int_{C_0^0[0,T-t_1]}\exp\{-\frac{\alpha}{2}{z_n}^2+\alpha(\beta q_n-g_n)z_n+\alpha\beta g_n q_n-\frac{\alpha}{2}{g_n}^2\}dm_p(y)$$

$$=\prod_{n=1}^{\infty}\left[\left\{\frac{1}{\sqrt{2\pi\alpha_n}}\int_{\mathbb{R}}\exp\{-\frac{\alpha}{2}u^2+\alpha\omega_n u-\frac{u^2}{2\alpha_n}\}du\right\}\cdot\exp\{\alpha\beta g_n q_n-\frac{\alpha}{2}g_n^2\}\right]$$

where $\omega_n=\beta q_n-g_n$. Hence the preceding equals

(3.9)
$$\prod_{n=1}^{\infty}\Big[\frac{1}{\sqrt{2\pi\alpha_n}}\exp\{-\frac{1}{2}(\alpha+\frac{1}{\alpha_n})(u^2-\frac{\alpha\alpha_n\omega_n}{\alpha\alpha_n+1})^2+\frac{\alpha^2\alpha_n\omega_n^2}{2(\alpha\alpha_n+1)}$$
$$+\alpha\beta g_n q_n-\frac{\alpha}{2}g_n^2\}\Big]$$
$$=\prod_{j=1}^{\infty}\Big[(1+\alpha\alpha_n)^{-\frac{1}{2}}\exp\Big\{\frac{\alpha^2\beta^2\alpha_n q_n^2}{2(\alpha\alpha_n+1)}+\frac{\alpha^2\alpha_n g_n^2}{2(\alpha\alpha_n+1)}-\frac{\alpha^2\beta\alpha_n g_n q_n}{\alpha\alpha_n+1}$$
$$+\alpha\beta g_n q_n-\frac{\alpha}{2}g_n^2\Big\}\Big]$$
$$=[\prod_{j=1}^{\infty}(1+\alpha\alpha_n)]^{-\frac{1}{2}}\exp\{\frac{\alpha^2\beta^2}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+1}q_n^2+\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+1}g_n^2$$
$$-\alpha^2\beta\sum_{n=1}^{\infty}\frac{\alpha_n}{1+\alpha\alpha_n}q_n g_n+\alpha\beta\sum_{n=1}^{\infty}g_n q_n-\frac{\alpha}{2}\sum_{n=1}^{\infty}g_n^2\}.$$

Using

$$\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2\pi^2})=\frac{\sin z}{z},$$

we have

$$\prod_{n=1}^{\infty}[1+\alpha\frac{(T-t_1)^2}{n^2\pi^2}]=\frac{\sinh\sqrt{\alpha}(T-t_1)}{\sqrt{\alpha}(T-t_1)}. \tag{3.10}$$

Observing that

(3.11)

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} q_n^2 = \int_0^{T-t_1} \int_0^{T-t_1} R(s,t,\alpha) q(s+t_1) q(t+t_1) ds dt$$

(3.12)

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} g_n^2 = \int_0^{T-t_1} \int_0^{T-t_1} R(s,t,\alpha) g(s) g(t) ds dt$$

(3.13)

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} q_n g_n = \int_0^{T-t_1} \int_0^{T-t_1} R(s,t,\alpha) q(s+t_1) g(t) ds dt$$

$$\sum_{n=1}^{\infty} g_n q_n = \int_0^{T-t_1} g(t) q(t+t_1) dt \tag{3.14}$$

$$\sum_{n=1}^{\infty} g_n^2 = \int_0^{T-t_1} g^2(t) dt \tag{3.15}$$

and using Lemmas 3.1 and 3.2 with replacing T by $T - t_1$, one can show that

(3. 16)

$$\frac{\alpha^2}{2} \sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha\alpha_n+1} g_n^2 - \frac{1}{2} \sum_{n=1}^{\infty} g_n^2$$

$$= -\frac{1}{2}\{\sqrt{\alpha}(\xi^2+\xi_1^2) \coth\sqrt{\alpha}(T-t_1) - \frac{2\sqrt{\alpha}\xi\xi_1}{\sinh\sqrt{\alpha}(T-t_1)} - \frac{(\xi-\xi_1)^2}{T-t_1}\}$$

(3.17)

$$-\alpha^2\beta \sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} g_n q_n + \alpha\beta \sum_{n=1}^{\infty} g_n q_n$$

$$= \alpha\beta(\xi-\xi_1) \int_0^{T-t_1} (\frac{\sinh\sqrt{\alpha}t}{\sinh\sqrt{\alpha}(T-t_1)} + \frac{\xi_1}{\xi-\xi_1}) q(t+t_1) dt.$$

Putting (3.10),(3.11),(3.16) and (3.17) in the last equation in (3.9) we obtain the desired result in the theorem.

COLLOARY 3.4. *Let α and F be as in Theorem 3.3. Let $0 = t_0 < t_1 < \cdots < t_n = T$. Then we have, for $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{R}^n$*

$$
\begin{aligned}
&E[F(x)|x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n] \\
&= \prod_{k=1}^{n} [(\frac{\sqrt{\alpha}(t_k - t_{k-1})}{\sinh\sqrt{\alpha}(t_k - t_{k-1})})^{\frac{1}{2}} \cdot \exp\{\frac{\xi_k - \xi_{k-1})}{2(t_k - t_{k-1})}\} \\
&\cdot \exp\{-\frac{\sqrt{\alpha}}{2}\coth\sqrt{\alpha}(t_k - t_{k-1})(\xi_k^2 + \xi_{k-1}^2) + \frac{\sqrt{\alpha}\xi_k\xi_{k-1}}{\sinh\sqrt{\alpha}(t_k - t_{k-1})}\} \\
&\cdot \exp\{\alpha\beta(\xi_k - \xi_{k-1})\int_0^{t_k - t_{k-1}} (\frac{\sinh\sqrt{\alpha}t}{\sinh\sqrt{\alpha}(t_k - t_{k-1})} \\
&\qquad + \frac{\xi_{k-1}}{\xi_k - \xi_{k-1}})q(t + t_{k-1})dt\} \\
&\cdot \exp\{\frac{\alpha^2\beta^2}{2}\int_0^{t_k - t_{k-1}}\int_0^{t_k - t_{k-1}} R(s,t,\alpha)q(s + t_{k-1})q(t + t_{k-1})dsdt\}],
\end{aligned}
$$

where $t_0 = 0, \xi_0 = 0$ and $R(s,t,\alpha)$ is as in (3.7) with replacing T by $t_k - t_{k-1}$.

Proof. Let $V(s,\xi) = \alpha\xi^2 - 2\alpha\beta q(s)\xi$. Since the Wiener process $\{x(s) : 0 \leq s \leq T\}$ is additive, it can be shown that

$$
\begin{aligned}
&E[\exp\{-\frac{1}{2}\int_0^T V(s,x(s))ds\}|x(t_k) = \xi_k, k = 1, 2, \cdots, n] \\
&\quad = E[\exp\{-\frac{1}{2}\sum_{k=1}^{n}\{\int_{t_{k-1}}^{t_k} V(s,x(s))ds\}\}|x(t_k) = \xi_k, k = 1, 2, \cdots, n] \\
&\quad = \prod_{k=1}^{n} E[\exp\{-\frac{1}{2}\int_{t_{k-1}}^{t_k} V(s,x(s))ds\}|x(t_{k-1}) = \xi_{k-1}, x(t_k) = \xi_k] \\
&\quad = \prod_{k=1}^{n} E[\exp\{-\frac{1}{2}\int_0^{t_k - t_{k-1}} W(s,x(s))ds\}|x(t_k - t_{k-1}) = \xi_k - \xi_{k-1}]
\end{aligned}
$$

where $W(s,x(s)) = V(s + t_{k-1}, x(s) + \xi_{k-1})$. Hence this, together with Theorem 3.3, gives the desired result.

If we let $q(t) \equiv 0$ in Corollary 3.4, we then have

COROLLARY 3.5. *Let α be a complex number with $Re\alpha > -\frac{\pi^2}{T^2}$. Let $0 = t_0 < t_1 < \cdots < t_n = T$. Then for $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{R}^n$,*

$$E[\exp\{-\frac{1}{2}\alpha \int_0^T x^2(s)ds\} \mid x(t_k) = \xi_k, k = 1, 2, \cdots, n]$$

$$= \prod_{k=1}^{n} [(\frac{\sqrt{\alpha}(t_k - t_{k-1})}{\sinh\sqrt{\alpha}(t_k - t_{k-1})})^{\frac{1}{2}} \cdot \exp\{\frac{\xi_k - \xi_{k-1}}{2(t_k - t_{k-1})}\}$$

$$\cdot\, exp\{-\frac{\xi_k^2 + \xi_{k-1}^2}{2}\sqrt{\alpha}\coth\sqrt{\alpha}(t_k - t_{k-1}) + \frac{\sqrt{\alpha}\xi_k\xi_{k-1}}{\sinh\sqrt{\alpha}(t_k - t_{k-1})}\}]$$

COROLLARY 3.6. *Let $Re\alpha > -\frac{\pi^2}{2T^2}$ and $\beta \in \mathbf{C}$. The function U defined on $\mathbf{R} \times [0, T] \times \mathbf{R}$*

(3.19)

$$U(\xi, t; \xi_0) = \sqrt{\frac{\sqrt{\alpha}csch\sqrt{\alpha}t}{2\pi}}\exp\{-\frac{\sqrt{\alpha}}{2}(\xi^2 + {\xi_0}^2)\coth\sqrt{\alpha}t + \frac{\sqrt{\alpha}\xi\xi_0}{\sinh\sqrt{\alpha}t}\}$$

$$\cdot\, exp\{\alpha\beta(\xi - \xi_0)\int_0^t (\frac{\sinh\sqrt{\alpha}s}{\sinh\sqrt{\alpha}t} + \frac{\xi_0}{\xi - \xi_0})q(s)ds\}$$

$$\cdot\, exp\{\frac{\alpha^2\beta^2}{2}\int_0^t\int_0^t R(s, \tau, \alpha)q(s)q(\tau)dsd\tau\}$$

is the solution of the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2} - \frac{\alpha}{2}\xi^2 U + \alpha\beta q(t)U \qquad (3.20)$$

satisfying the condition $U(\xi,\, t; \xi_0) \to \delta(\xi - \xi_0)$ as $t \downarrow 0$.

Proof. From a theorem of Kac[4], the function

$$U(\xi, t; \xi_0) = E[\exp\{-\frac{\alpha^2}{2}\int_0^t x^2(s)ds + \alpha\beta\int_0^t q(s)x(s)dx\}|x(0) = \xi_0,$$

$$x(t) = \xi] \cdot \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(\xi - \xi_0)}{2t}\}$$

is the solution of the differential equation (3.20). So by Theorem 3.3, the function $U(\xi, t; \xi_0)$ in the corollary is the solution of the differential equation (3.20).

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