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A LITTLE GENERALIZATION OF HAHN–BANACH EXTENSION PROPERTY

SUNG HO PARK

Let M be a linear subspace of a normed linear space X and let V be a linear subspace of the dual space X^*. In [11], I. Singer gave some sufficient conditions for which M has the Hahn-Banach extension property in V. In [10], R.R. Phelps studied the unique Hahn-Banach extension property. In this paper, we are interested in a sufficient and necessary condition for which M has the (unique) Hahn-Banach extension property in V by using best approximations and its applications. Here, first we give the definition of the Hahn-Banach extension property in V.

DEFINITION 1. Let M be a linear subspace of a normed linear space X, and V a linear subspace of the dual space X^*. We say that M has the Hahn-Banach extension property in V if for each $f \in V$ there exists $f_0 \in V$ such that

(1) $f_0(x) = f(x)$ for each $x \in M$, and
(2) $\|f_0\| = \|f|_M\|$.

Here we give some examples which has the Hahn-Banach extension property and which does not have the Hahn-Banach extension property.

EXAMPLES 2. (1) Let $X = \mathbf{R}^3, M = [(1,1,0)]$, and $V = [(0,1,2)]$ with the usual norm, where $[x]$ denotes the subspace generated by x. Then $M^\perp = [(1,-1,0),(0,0,1)]$ and $M^\perp \cap V = \{0\}$. If $f = (0,1,2)$, then clearly $\|f\| = \sqrt{5}$, and $\|f|_M\| = 1/\sqrt{2}$. By Theorem 5, there exist no the Hahn-Banach extensions of $f|_M$ in V.

More generally, we can choose a linear subspace M of a normed linear space X and a linear subspace V of the dual space X^* which satisfy the following conditions:
(i) $M^\perp \cap V = \{0\}$,

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(ii) there exists $f \in V$ such that $\|f|_M\| \neq \|f\|$.
In this case, M does not have the Hahn-Banach extension property in V.

(2) Let $X = \mathrm{R}^3$, $M = [(1,0,0),(0,1,0)]$ and $V = [(0,1,0),(0,0,1)]$ with the usual norm. Then clearly $M^\perp \cap V = [(0,0,1)]$. Let $f = (0,x,y)$ in V. Then $f|_M : (u,v,0) \longrightarrow xv$. Put $f_0 = (0,x,0)$. Then $f_0 \in V$, and $f|_M = f_0$ on M. Moreover, $d(f, M^\perp \cap V) = |x| = \|f\ |_M\ \|$ and clearly $M^\perp \cap V$ is proximinal in V. Therefore, M has the Hahn-Banach extension property in V.

Throughout this paper, let M be a linear subspace of a normal linear space $X, M^\perp$ the annihilator in the dual space X^*, that is,

$$M^\perp = \{f \in X^* : f(m) = 0 \quad \text{for every} \quad m \in M\},$$

and $M_V^\perp$ the annihilator in a subspace V of the dual space X^*, that is,

$$M_V^\perp = \{f \in V : f(m) = 0 \quad \text{for every} \quad m \in M\}.$$

Further, let $J : X \longrightarrow X^{**}$ denote the cannonial embedding of X into its second dual $X^{**} : J(x) = \hat{x}$, where $\hat{x}(f) = f(x), f \in X^*$.

LEMMA 3 [6],[11]. *Let M be a linear subspace of a normed linear space X. Then for each $f \in X^*$,*

$$d(f, M^\perp) = \|f|_M\|.$$

In particular, $M^\perp$ is proximinal in X^*.

Proof. If $g \in M^\perp$, then

$$\begin{aligned}\|f|_M\| &= \sup\{|(f-g)(x)| : x \in M, \|x\| \leq 1\} \\ &\leq \|f - g\|,\end{aligned}$$

so $\|f|_M\| \leq d(f, M)$. On the other hand, by the Hahn-Banach theorem, we can choose $h \in X^*$ such that $h = f$ on M and $\|h\| = \|f|_M\|$. Then $f - h \in M^\perp$ and $\|f|_M\| = \|f - (f-h)\| \geq d(f, M^\perp)$. Therefore, $d(f, M^\perp) = \|f|_M\|$.

: We recall the following well-known results ([2],[11]) which we shall use in the sequel.

LEMMA 4. *Let X be a normed linear space and V a total linear subspace of the dual space X^*. Then*

(a) *a linear subspace M of X is $\sigma(X, V)$-closed if and only if for each $x \notin M$ there exists $f \in M^\perp \cap V$ with $f(x) = 1$.*

(b) *every finite-dimensional subspace M of X is $\sigma(X, V)$-closed.*

(c) *if M is a $\sigma(X, V)$-closed linear subspace of X and G is a finite-dimensional subspace of X such that $M \cap G = \{0\}$, then $M \oplus G$ is $\sigma(X, V)$-closed.*

Now we give a sufficient and necessary condition for Hahn-Banach extension property.

THEOREM 5. *Let M be a linear subspace of a normed linear space X, and V a linear subspace of the dual space X^*. Then the following statements are equivalent:*

(a) *M has the Hahn-Banach extension property in V.*

(b) *(i) $M_V^\perp$ is proximinal in V,*
(ii) for each $f \in V$, $d(f, M_V^\perp) = \|f|_M\|$.

Proof. $(a) \longrightarrow (b)$ Suppose that (a) holds, that is, for each $f \in V$, there exists an element $f_0 \in V$ such that $f_0(x) = f(x)$ for each $x \in M$ and $\|f_0\| = \|f|_M\|$. Then $f - f_0 \in M_V^\perp$ and $d(f, M_V^\perp) \le \|f-(f-f_0)\| = \|f|_M\|$. Since clearly $\|f|_M\| \le d(f, M_V^\perp)$, $d(f, M_V^\perp) = \|f|_M\|$. Thus (a) implies (b).

$(b) \longrightarrow (a)$ Suppose that $M_V^\perp$ is proximinal in V and that for each $f \in V$, $d(f, M_V^\perp) = \|f|_M\|$. Let f be a fixed element of V. Since $M_V^\perp$ is proximinal in V, there exists an element g in $M_V^\perp$ such that $\|f - g\| = d(f, M_V^\perp) = \|f|_M\|$. Since $g \in M_V^\perp$ and $f \in V$, $f - g \in V$, $(f - g)(x) = f(x) (x \in M)$ and $\|f - g\| = \|f|_M\|$. Therefore (b) implies (a).

COROLLARY 6. *Let X be a normed linear space, M a linear subspace of X and V a linear subspace of X^*, such that $M^\perp \subset V$. Then M has the Hahn-Banach extension property in V.*

Proof. It follows from Lemma 3 and Theorem 5.

REMARK. Corollary 6 was proven in [11, Proposition 2].

COROLLARY 7. *Let M be a linear subspace of a normed linear space X. Then $M^\perp$ is proximinal in X^* and $d(f, M^\perp) = \|f|_M\|$ for each $f \in X^*$.*

Proof. It follows from the Hahn-Banach Theorem.

LEMMA 8. *Let X be a normed linear space, V a total linear subspace of X^*, and M a $\sigma(X,V)$-closed subspace of finite codimension in X. Then $M^\perp \subset V$.*

Proof. Since M is also norm-closed, let $\{x_i\}_1^n \subset X$ be linearly independent such that $M \oplus [x_i]_{i=1}^n = X$. Then, since M is $\sigma(X,V)$-closed and $\dim[x_i]_{i\neq j} < \infty$, the subspace $M \oplus [x_i]_{i\neq j} (j = 1, 2, \cdots, n)$ are $\sigma(X,V)$-closed [Lemma 4,(c)]. Hence, since $x_j \notin M \oplus [x_i]_{i\neq j}$, there exists (by Lemma 4.(a)) $f \in M^\perp \cap V (i = 1, 2, \cdots, n)$ such that $f_i(x_j) = \delta_{ij} (i, j = 1, 2, \cdots, n)$. But then $f_1, f_2, \cdots, f_n$ are independent, so $\dim[f_i]_{i=1}^n = n$, whence since $[f_i]_{i=1}^n \subset M^\perp$ and $\dim M^\perp = n$, so we obtain $[f_i]_{i=1}^n = M^\perp$. Consequently, $M^\perp = [f_i]_{i=1}^n \subset V$.

REMARK. The proof of Lemma 8 also can be found in the proof of [11, Proposition 3].

COROLLARY 9. *Let X be a normed linear space, V a total linear subspace of X^*, and M a $\sigma(X,V)$-closed subspace of finite codimension in X. Then M has the Hahn-Banach extension property in V.*

Proof. It follows from Corollary 6 and Lemma 8.

It is well-known that if $M^\perp$ or V has finite dimension, then $M_V^\perp$ is proximinal in V, so we can have the following property.

COROLLARY 10. *If $M^\perp$ or V has finite dimension, then the following statements are equivalent:*
(a) M has the Hahn-Banach extension property in V.
(b) $d(f, M_V^\perp) = \|f|_M\|$ for each $f \in V$.

Proof. Since in either cases $M_V^\perp$ is proximinal in V, it follows from Theorem 5.

COROLLARY 11. *Let X be a normed linear space X and M a linear subspace of X^*. Then the following statements are equivalent:*
(a) M has the Hahn-Banach extension property in $J(X)$.
(b) i) $M_\wedge^\perp$ is proximinal in $J(X)$ where $M_\wedge^\perp = \{\hat{x} \in J(X) : \hat{x}(f) = 0,$ for all $f \in V\}$
ii) For each $x \in X$, $d(\hat{x}, M_\wedge^\perp) = \|\hat{x}\|_M$.

Proof. It follows from Theorem 5.

COROLLARY 12. *Let X be a normed linear space and M a linear subspace of X^* such that $M^\perp \subset J(X)$, where $J : X \longrightarrow X^{**}$ is the cannonial embedding. Then M has the Hahn-Banach extension property in $J(X)$. That is, for each $x \in X$ there exists an element $x_0 \in X$ such that*
(1) $f(x_0) = f(x)$ for each f in M,
(2) $\|x_0\| = sup\{|f(x)| : f \in M, \|f\| \leq 1\}$.

In particular, if M is a $\sigma(X^*, X)$-closed linear subspace of finite codimension in X^*, then for every $x \in X$ there exists $x_0 \in X$ satisfying (1) and (2).

Proof. It follows from Corollary 9 and Corollary 11.

DEFINITION 13. Let M be a linear subspace of a normed linear space X, and V a linear subspace of X^*. We say that M has the unique Hahn-Banach extension property in V or the property U in V if for each $f \in V$ there exists a unique element $f_0 \in V$ such that
(1) $f_0(x) = f(x)$ for each $x \in M$, and
(2) $\|f_0\| = \|f|_M\|$.

REMARK. In [10], R.R. Phelps defined and studied the unique Hahn-Banach extension property or the property U.

Now we give a sufficient and necessary condition for which M has the Hahn-Banach extension property in V.

THEOREM 14. *Let M be a linear subspace of a normed linear space X, and V a linear subspace of X^*. Then the following statements are equivalent:*
(a) M has the unique Hahn-Banach extension property in V.

(b) $M_V^{\perp}$ *is Chebyshev in* V *and for each* $f \in V, d(f, M_V^{\perp}) = \|f|_M\|$.

Proof. $(a) \longrightarrow (b)$ Suppose that M has the unique Hahn-Banach extension property in V, that is, for each $f \in V$ there exists unique $f_0 \in V$ such that $f_0(x) = f(x)(x \in M)$ and $\|f_0\| = \|f|_M\|$. Then $f - f_0 \in M_V^{\perp}$, and $\|f|_M\| \leq d(f, M_V^{\perp}) \leq \|f_0\| = \|f|_M\|$. Then $d(f, M_V^{\perp}) = \|f|_M\|$. Since f_0 is unique, $M_V^{\perp}$ is Chebyshev in V.

$(b) \longrightarrow (a)$ Suppose that (b) holds. Then for each $f \in V$ there exists unique $g_0 \in M_V^{\perp}$ such that $\|f - g_0\| = \|f|_M\|$. Let $f_0 = f - g_0$. Then $f_0 \in V$, $f_0(x) = f(x)(x \in M)$ and $\|f_0\| = \|f|_M\|$. Thus (a) holds.

COROLLARY 15 [10]. *A linear subspace* M *of* X *has the unique Hahn-Banach extension property in* X^* *if and only if its annihilator* $M^{\perp}$ *is Chebyshev in* X^*.

Proof. It follows from Lemma 3 and Theorem 14.

COROLLARY 16 [10]. *If* X *is a reflexive space, then a closed linear subspace* M *of* X *is Chebyshev if and only if* $M^{\perp}$ *has the unique Hahn-Banach extension property in* X^{**}.

Proof. It follows from Theorem 14.

Now we reduce results relating approximative property of $M_V^{\perp}$ with properties of extending continuous linear functionals in $V|_M$ to elements of V.

DEFINITION 17. For a linear subspace M of X, a linear subspace V of X^* and m^* in $V|_M$, let $N_M^V(m^*)$ denote the set of all Hahn-Banach extensions of m^* in V; that is,

$$N_M^V(m^*) = \{f \in V | f|_M = m^*, \|f\| = \|m^*\|\}.$$

REMARK. $N_M^V(m^*)$ may be empty. But if M has the Hahn-Banach extension property in V, then N_M^V is nonempty, so $N_M^V : V|_M \longrightarrow 2^V \backslash \{\emptyset\}$.

THEOREM 18. *Let M be a linear subspace of X and V a linear subspace of the dual space X^*. If M has the Hahn-Banach extension property in V, then for each $f \in V$,*

$$P_{M_V^\perp}(f) = f - N_M^V(f|_M).$$

Proof. Since M has the Hahn-Banach extension property in V, $N_M^V(f|_M) \neq \emptyset$ for each $f \in V$. Then, for each $f \in V$,

$$\begin{aligned} g \in N_M^V(f|_M) &\leftrightarrow g \in V, g|_M = f|_M \text{ and } \|g\| = \|f|_M\| \\ &\leftrightarrow m^* := f - g \in M^\perp \text{ and } \|g\| = d(f, M_V^\perp) \\ &\leftrightarrow \|g\| = \|f - m^*\| = d(f, M_V^\perp) \\ &\leftrightarrow m^* = f - g \in P_{M_V^\perp}(f). \end{aligned}$$

Thus for each $f \in V$,

$$P_{M_V^\perp}(f) = f - N_M^V(f|_M).$$

DEFINITION 19. Let Y be a metric space, $F : X \longrightarrow 2^Y$, and $x_0 \in X$. Then F is called:

(1) upper semicontinuous (u.s.c.) at x_0 if for any set $V \supset F(x_0)$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for each $x \in U$;

(2) lower semicontinuous (l.s.c.) at x_0 if for any set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for each $x \in U$;

(3) upper Hausdorff semicontinuous (u.H.s.c.) at x_0 if for each $\epsilon > 0$ there exists a neighborhood U of x_0 such that $F(x) \subset B_\epsilon(F(x_0)) := \{y \in Y : d(y, F(x_0)) < \epsilon\}$ for each $x \in U$;

(4) lower Hausdorff semicontinuous (l.H.s.c.) at x_0 if for each $\epsilon > 0$ there exists a neighborhood U of x_0 such that $F(x_0) \subset B_\epsilon(F(x))$ for each $x \in U$.

For equivalent formulations of these properties, as well as relationship holding between them, see. e.g. [6].

LEMMA 20 [6]. *Let M be a proximinal in $X, x_0 \in X$, and $\tau = u, l, u.H., l.H$. Then P_M is τ.s.c. at x_0 if and only if $I - P_M$ is τ.s.c. at x_0.*

THEOREM 21. *Let M be a linear subspace of a normed linear space X which has the Hahn-Banach extension property in a linear subspace V of the dual space $X^*, f \in V$ and $\tau = u, l, l.H., u.H$. Then $P_{M_V^\perp}$ is τ.s.c. at f if and only if N_M^V is τ.s.c. at $f|_M$.*

Proof. Suppose that $P_{M_V^\perp}$ is u.s.c. at f and that W is an open set with $W \supset N_M^V(f|_M)$. By Theorem 18, $W \supset (1 - P_{M_V^\perp})(f)$. By Lemma 20, $I - P_{M_V^\perp}$ is u.s.c. at f so there exists a neighborhood U of f such that $(I - P_{M_V^\perp})(g) \subset W$ for all $g \in U$. Thus $N_M^V(g|_M) \subset W$ for all $g \in U$. Then $U|_M$ is a neighborhood of $f|_M$ in $V|_M$ and $N_M^V(g|_M) \subset W$ for all $g|_M \in U|_M$. Thus N_M^V is u.s.c. at $f|_M$.

Conversely, if N_M^V is u.s.c. at $f|_M$, let W be open and $W \supset (I - P_{M_V^\perp})(f) = N_M^V(f|_M)$. Select a neighborhood $\tilde{U}$ of $f|_M$ in $V|_M$ such that $N_M^V(g|_M) \subset W$ for all $g|_M \in \tilde{U}$. Since $R_M : V \longrightarrow V|_M$, defined by $R(f) = f|_M$, is continuous, the set $U = R_M^{-1}(\tilde{U})$ is open in V and contains f. Moreover, for each $g \in U$, $g|_M \in \tilde{U}$ and $(I - P_{M_V^\perp})(g) = N_M^V(g|_M) \subset W$. Thus $I - P_{M_V^\perp}$ is u.s.c. at f. By Lemma 20, $P_{M_V^\perp}$ is u.s.c. at f.

The proofs when $\tau = 1, 1.H.$, or u.H. are similar.

The next theorem shows that the existence of continuous, Lipschitz continuous, or linear selection for $P_{M_V^\perp}$ is equivalent to analogous property for N_M^V. (Recall that a selection for the set-valued mapping $F : X \longrightarrow 2^Y$ is any function $f : X \longrightarrow Y$ with $f(x) \in F(x)$ for all x. Moreover, a selection p for P_M is called additive modulo M if $p(x + y) = p(x) + y$ whenever $x \in X, y \in M$).

THEOREM 22. *Let M be a linear subspace of a normed linear space X which has the Hahn-Banach extension property in a linear subspace V of the dual space X^*.*

(1) $P_{M_V^\perp}$ has a continuous (resp. linear) selection if and only if N_M^V has a continuous (resp. linear) selection.

(2) $P_{M_V^\perp}$ has a linear selection if and only if N_M^V has a linear selection with norm one.

(3) $P_{M_V^\perp}$ has a Lipschitz (resp. pointwise Lipschitz) continuous selection which is additive modulo $M_V^\perp$ if and only if N_M^V has a Lipschitz (resp. pointwise Lipschitz) continuous selection.

Proof. (1) If $P_{M_V^\perp}$ has a continuous selection, then it has a continuous selection p which is also homogeneous and additive modulo $M_V^\perp$ [3, Theorem 3.4]. Define e on $V|_M$ by $e(f|_M) = f - p(f)$. Then e is well-defined since if $f|_M = g|_M$, then $m = f - g \in M_V^\perp$ and $f - p(f) = g + m - p(g + m) = g - p(g)$. Moreover, by Theorem 18, e is a selection for N_M^V. Now if $f|_M$ and $g|_M$ are in $V|_M$, then there exists $h \in V$ such that $\|(f - g)|_M\| = \|h\| = \|f - (f - h)\|$ and $g|_M = (f - h)|_M$ since M has the Hahn-Banach extension property in V. Then

$$\begin{aligned}\|e(f|_M) - e(g|_M) &= \|e(f|_M) - e((f - h)|_M)\| \\ &= \|f - p(f) - (f - h - p(f - h))\| \\ &\leq \|h\| + \|p(f) - p(f - h)\| \\ &= \|(f - g)|_M\| + \|p(f) - p(f - h)\|.\end{aligned}$$

Since p is continuous at f, given any $\epsilon > 0$, choose $0 < \delta < \epsilon$ such that $\|f - g\| < \delta$ implies $\|p(f) - p(g)\| < \epsilon$. Thus, if $g \in V$ is chosen so that $\|f|_M - g|_M\| < \delta$, then $\|f - (f - h)\| < \delta$ so that $\|e(f|_M) - e(g|_M)\| < 2\epsilon$. This proved that e is continuous at $f|_M$.

Conversely, suppose that N_M^V has a continuous selection e. Define p on V by $p(f) = f - e(f|_M)$. Then p is a selection for $P_{M_V^\perp}$ by Theorem 18. Given $\epsilon > 0$ and $f \in V$, choose $0 < \delta < \epsilon$ so that $\|e(f|_M) - e(g|_M)\| < \epsilon$ whenever $\|f|_M - g|_M\| < \delta$. Thus if $\|f - g\| < \delta$, then $\|f|_M - g|_M\| \leq \|f - g\| < \delta$ so that

$$\begin{aligned}\|p(f) - p(g)\| &\leq \|f - g\| + \|e(f|_M) - e(g|_M)\| \\ &< \delta + \epsilon < 2\epsilon.\end{aligned}$$

Thus p is continuous at f.

The proof that $P_{M_V^\perp}$ has a linear selection if and only if N_M^V has a linear selection is similar.

(3) Suppose that p is a pointwise Lipschitz continuous selection for $P_{M_V^\perp}$ which is additive modulo $M_V^\perp$. Then, just as in the proof of (1), the function e defined on $V|_M$ by $e(f|_M) = f - p(f)$ is a selection for N_M^V. Moreover, given $f|_M \in V|_M$ and $g|_M \in V|_M$, there exists $h \in V$ such that $\|(f-g)|_M\| = \|h\| = \|f-(f-h)\|$ and $g|_M = (f-h)|_M$ since M has the Hahn-Banach extension property in V. Thus

$$\begin{aligned}
\|e(f|_M) - e(g|_M) &= \|e(f|_M) - e((f-h)|_M)\| \\
&= \|f - p(f) - (f - h - p(f-h))\| \\
&\leq \|h\| + \|p(f) - p(f-h)\| \\
&\leq \|h\| + \lambda(f)\|h\| \\
&= (1+\lambda(f))\|f|_M - g|_M\|.
\end{aligned}$$

Thus e is pointwise Lipschitz continuous at $f|_M$ with Lipschitz constant $1+\lambda(f)$.

Conversely, let e be a pointwise Lipschitz continuous selection for N_M^V. Defining p on V by $p(f) = f - e(f|_M)$, we see that p is a selection for $P_{M_V^\perp}$ such that for every $f \in V$ and $m \in M_V^\perp$

$$\begin{aligned}
p(f+m) &= f + m - e((f+m)|_M) \\
&= f + m - e(f|_M) = p(f) + m.
\end{aligned}$$

That is, p additive modulo $M_V^\perp$. Then

$$\begin{aligned}
\|p(f) - p(g)\| &\leq \|f-g\| + \|e(f|_M) - e(g|_M)\| \\
&\leq \|f-g\| + \lambda(f|_M)\|f|_M - g|_M\| \\
&\leq (1+\lambda(f|_M))\|f-g\|.
\end{aligned}$$

This shows that p is pointwise Lipschitz continuous at f with Lipschitz constant $1+\lambda(f|_M)$.

The proof of the global Lipschitz properties now follows immediately since in this case the Lipschitz constants are independent of the particular points.

COROLLARY 23 [5]. *Let M be a linear subspace of a normed linear space X which has the Hahn-Banach extension property in a linear subspace V of the dual space X^*. Suppose that $M_V^\perp$ is complemented in V. Then $P_{M_V^\perp}$ has a Lipschitz (resp. pointwise Lipschitz) continuous selection if and only if N_M^V has Lipschitz (resp. pointwise Lipschitz) continuous selection.*

Proof. In [3], it was shown that, when $M_V^\perp$ is complemented, $P_{M_V^\perp}$ has a Lipschitz (resp. pointwise Lipschitz) continuous selection if and only if $P_{M_V^\perp}$ has one which is also homogeneous and additive modulo $M_V^\perp$. An appeal to Theorem 22 completes the proof.

COROLLARY 24 [5]. *Let M be a linear subspace of a normed linear space X. Then*

(1) For each $f \in X^*$,

$$P_{M^\perp}(f) = f - N_M(f|_M).$$

where $N_M(f|_M) = \{f_0 \in X^* | f_0|_M = f|_M, \|f_0\| = \|f|_M\|\}$.

(2) $P_{M^\perp}$ is τ.s.c. at f if and only if N_M is τ.s.c. at $f|_M$. (Here, $\tau = 1, u, l.H., u.H.$)

(3) $P_{M^\perp}$ has a continuous (resp. linear) selection if and only if N_M has a continuous (resp. linear) selection.

(4) $P_{M^\perp}$ has a linear selection if and only if N_M has a linear selection with norm one.

(5) $P_{M^\perp}$ has a Lipschitz (resp. pointwise Lipschitz) continuous selection which is additive modulo $M^\perp$ if and only if N_M has a Lipshitz (resp. pointwise Lipschitz) continuous selection.

Proof. By Hahn-Banach theorem, M has the Hahn-Banach extension property in X^*. Thus it follows from Lemma 20 and theorem 22.

COROLLARY 25 [5]. *Let M be a subspace of X whose annihilator $M^\perp$ is comlemented. Then $P_{M^\perp}$ has a Lipschitz (resp. pointwise Lipschitz) continuous selection if and only if N_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection.*

Proof. If follows from Corollary 23.

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Sogang University
Department of Mathematics
CPO 1142
Seoul, 121-742, Korea