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SWICHING RULE ON THE SHLFEDE RIM HOOK TABLEAUX

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0. Introduction.

When the Schur function s_λ corresponding to a partition λ is defined as the generating function of the column strict tableaux of shape λ it is not at all obvious that s_λ is symmetric. In [BK] Bender and Knuth showed that s_λ is symmetric by describing a switching rule for column strict tableaux, which is essentially equivalent to the jeu de taquin of Schützenberger (see [Sü]). Bender and Knuth's switching rule shows that the number of column strict tableaux of a given shape is independent of the order of the contents. Stanton and White[SW2] gave a rim hook analog of this switching procedure.

In this paper we describe a switching algorithm for shifted rim hook tableaux, which shows that the sum of the weights of the shifted rim hook tableaux of a given shape and content does not depend on the order of the content if content parts are all odd. Using the recurrence formula for the irreducible spin characters of $\tilde{S}_n$, this will show that $\varphi^\lambda_\rho = \varphi^\lambda_{\rho'}$, where ρ has all odd parts and ρ' is any reordering of ρ.

In section 1, we outline the definitions and notation used in this paper. In section 2, we review the basic properties of a group $\tilde{S}_n$ and draw some relations between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions. In section 3, a swiching rule on the shifted rim hook tableaux is given.

1. Definitions

We use standard notation **P**, **Z**, **Q**, **C** for the set of all positive integers, the ring of integers, the field of rational numbers and the field of complex numbers, respectively.

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DEFINITION 1.1. A *partition* λ of a nonnegative integer n is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
(2) $\sum_{i=1}^{\ell} \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$. We say each term λ_i is a *part* of λ and n is the *weight* of λ. The number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. Let $\mathcal{P}$ be the set of all partitions and $\mathcal{P}_n$ be the set of all partitions of n.

We sometimes abbreviate the partition λ with the notation $1^{j_1}2^{j_2}\dots$, where j_i is the number of parts of size i. Sizes which do not appear are omitted and if $j_i = 1$, then it is not written. Thus, a partition $(5,3,2,2,2,1) \vdash 15$ can be written $12^3 35$.

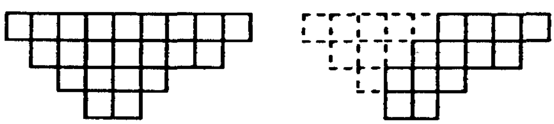

Figure 1.1

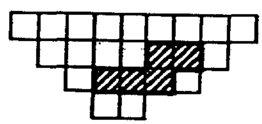

Figure 1.2

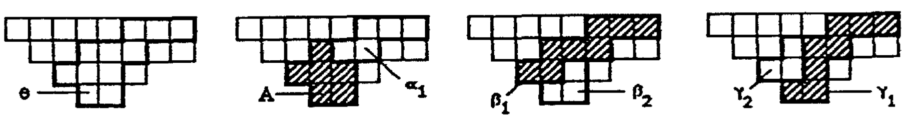

Figure 1.3

NOTATION 1.2. We denote

$$\begin{aligned} OP_n &= \{\mu \in \mathcal{P}_n \mid \text{every part of } \mu \text{ is odd}\}, \\ DP_n &= \{\mu \in \mathcal{P}_n \mid \mu \text{ has all distinct parts}\}, \\ DP_n^+ &= \{\mu \in DP_n \mid n - \ell(\mu) \text{ is even}\} \quad \text{and} \\ DP_n^- &= \{\mu \in DP_n \mid n - \ell(\mu) \text{ is odd}\}. \end{aligned}$$

DEFINITION 1.3. For each $\lambda \in DP$, a *shifted diagram* D'_λ of shape λ is defined by

$$D'_\lambda = \{(i,j) \in \mathbf{Z}^2 \mid i \leq j \leq \lambda_j + i - 1, 1 \leq i \leq \ell(\lambda)\}.$$

And for $\lambda, \mu \in DP$ with $D'_\mu \subseteq D'_\lambda$, a *shifted skew diagram* $D'_{\lambda/\mu}$ is defined as the set-theoretic difference $D'_\lambda \setminus D'_\mu$. *Figure 1.1* shows D'_λ and $D'_{\lambda/\mu}$ respectively when $\lambda = (9,7,4,2)$ and $\mu = (5,3,1)$.

DEFINITION 1.4. A shifted skew diagram θ is called a *single rim hook* if θ is connected and contains no 2×2 block of cells. If θ is a single rim hook, then its ***head*** is the upper rightmost cell in θ and its *tail* is the lower leftmost cell in θ. See *Figure 1.2.*

DEFINITION 1.5. A ***double rim hook*** is a shifted skew diagram θ formed by the union of two single rim hooks both of whose tails are on the main diagonal. If θ is a double rim hook, we denote by $\mathcal{A}[\theta]$ (resp., $\alpha_1[\theta]$) the set of diagonals of length two (resp., one). Also let $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) be a single rim hook in θ which starts on the upper (resp., lower) of the two main diagonal cells and ends at the head of $\alpha_1[\theta]$. The tail of $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) is called the ***first tail*** (resp., ***second tail***) of θ and the head of $\beta_1[\theta]$ or $\gamma_1[\theta]$ (resp., $\gamma_2[\theta], \beta_2[\theta]$, where $\beta_2[\theta] = \theta \setminus \beta_1[\theta]$ and $\gamma_2[\theta] = \theta \setminus \gamma_1[\theta]$) is called the ***1st head*** (resp., ***second head, third head***) of θ. Hence we have the following descriptions for a double rim hook θ:

$$\begin{aligned}\theta &= \mathcal{A}[\theta] \cup \alpha_1[\theta] \\ &= \beta_1[\theta] \cup \beta_2[\theta] \\ &= \gamma_1[\theta] \cup \gamma_2[\theta].\end{aligned}$$

Definition 1.5 is illustrated in *Figure 1.3.* We write $\mathcal{A}, \alpha_1$, etc. for $\mathcal{A}[\theta], \alpha_1[\theta]$, etc. when there is no confusion.

We will use the term ***rim hook*** to mean a single rim hook or a double rim hook.

DEFINITION 1.6. A ***shifted rim hook tableau*** of shape $\lambda \in DP$ and content $\rho = (\rho_1, \ldots, \rho_m)$ is defined recursively. If $m = 1$, a rim hook with all 1's and shape λ is a shifted rim hook tableau. Suppose P of shape λ has content $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the cells containing the m's form a rim hook inside λ. If the removal of the m's leaves a shifted rim hook tableau, then P is a shifted rim hook tableau. We define a ***shifted skew rim hook tableau*** in a similar way.

DEFINITION 1.7. If θ is a single rim hook then the *rank* $r(\theta)$ is one less than the number of rows it occupies and the *weight* $w(\theta) = (-1)^{r(\theta)}$; if θ is a double rim hook then the *rank* $r(\theta)$ is $|\mathcal{A}[\theta]|/2 + r(\alpha_1[\theta])$ and the *weight* $w(\theta)$ is $2(-1)^{r(\theta)}$.

The *weight* of a shifted rim hook tableau P, $w(P)$, is the product of the weights of its rim hooks. The weight of a shifted skew rim hook tableau is defined in a similar way.

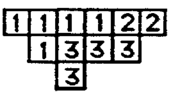

Figure 1.4

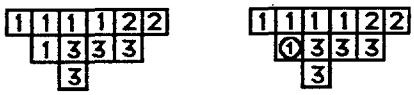

Figure 1.5

Let P be a shifted rim hook tableau. We write $\kappa_P\langle r\rangle$ (or just $\kappa\langle r\rangle$) for a rim hook of P containing r. *Figure 1.4* shows an example of a shifted rim hook tableau P of shape $(6,4,1)$ and content $(5,2,4)$. Here $r(\kappa\langle 1\rangle) = 1$, $r(\kappa\langle 2\rangle) = 0$ and $r(\kappa\langle 3\rangle) = 1$. Also $w(\kappa\langle 1\rangle) = -2$, $w(\kappa\langle 2\rangle) = 1$ and $w(\kappa\langle 3\rangle) = -1$. Hence $w(P) = (-2)\cdot(1)\cdot(-1) = 2$.

DEFINITION 1.8. Suppose P is a shifted rim hook tableau. Then we denote by P_2 one of the tableaux obtained from P by circling or not circling the second tail of each double rim hook in P. The P_2 is called a *second tail circled rim hook tableau.* We use the notation $|\cdot|$ to refer to the uncircled version; e.g., $|P_2| = P$. See *Figure 1.5* for examples of second tail circled rim hook tableaux.

We now define a new weight function w' for second tail circled rim hook tableaux. If τ is a rim hook of P_2, we define $w'(\tau) = (-1)^{r(\tau)}$. The weight $w'(P_2)$ is the product of the weights of rim hooks in P_2.

For each double rim hook τ of a rim hook tableau P, there are two second circled rim hooks τ_1, τ_2 such that $w(\tau) = w'(\tau_1) + w'(\tau_2)$. This fact implies the following:

PROPOSITION 1.9. *Let $\gamma \in OP$. Then we have*

$$\sum_P w(P) = \sum_{P_2} w'(P_2),$$

where the left-hand sum is over all shifted rim hook tableaux P of shape λ/μ and content γ, while the right-hand sum is over all shifted second tail circled rim hook tableaux P_2 of shape λ/μ and content γ.

2. Symmetric functions and irreducible spin characters of $\tilde{S}_n$.

We consider the ring $\mathbf{Z}[x_1, x_2, \ldots]$ of formal power series with integer coefficients in the infinite variables $x_1, x_2, \cdots$. Note that the symmetric functions form a subring of $\mathbf{Z}[x_1, x_2, \ldots]$. Let $\Lambda(x)$, or simply Λ, be the ring of symmetric functions of $x_1, x_2, \cdots$. Define $\mathbf{Z}$-modules Λ^k by $\Lambda^k(x) = \Lambda^k = \{f \in \Lambda \mid f \text{ is homogeneous of degree } k\}$. Then we have $\Lambda = \prod_{k \geq 0} \Lambda^k$.

DEFINITION 2.1. Let r be a positive integer. The rth *power sum* p_r is defined by

$$p_r = \sum_{i \geq 1} x_i^r.$$

By convention, we set $p_0 = 1$ and $p_r = 0$ for $r < 0$. Extend the definition of this symmetric function to all partitions by $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$.

We now define a group $\tilde{S}_n$ and draw some connections between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions.

DEFINITION 2.2. For $n > 1$ let $\tilde{S}_n$ be the group generated by $t_1, t_2, \ldots, t_{n-1}, -1$ subject to relations

$$\begin{gathered} t_i^2 = -1 \quad \text{for } i = 1, 2, \ldots, n-1, \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, \\ t_i t_j = -t_j t_i \quad \text{for } |i - j| > 1 \ (i, j = 1, 2, \ldots, n-1). \end{gathered}$$

Note that $|\tilde{S}_n| = 2n!$. Since -1 is a central involution, Schur's lemma implies that an irreducible representation of $\tilde{S}_n$ must represent -1 by either the scalar 1 or -1. The representation of the former type is an ordinary representation of S_n, while one of the latter type will correspond to a projective representation of S_n, as we will see later. A representation T of $\tilde{S}_n$ is called a *spin representation* of $\tilde{S}_n$ if the group element -1 is represented by scalar -1, i.e., if $T(-1) = -1$.

To describe the characters of spin representations of $\tilde{S}_n$ we consider the structure of the conjugacy classes of $\tilde{S}_n$. Let $\theta_n : \tilde{S}_n \to S_n$ be an epimorphism defined by $t_i \mapsto s_i$, where s_i is an adjacent transposition $(i\ i+1)$ in S_n. For each partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of n, we choose a specific element σ_μ such that $\theta_n(\sigma_\mu)$ is of cycle-type μ as follows: Define

$$\sigma_\mu = \pi_1 \pi_2 \ldots \pi_\ell,$$

where $\pi_j = t_{a+1} t_{a+2} \ldots t_{a+\mu_j - 1}$ $(a = \sum_{i=1}^{j-1} \mu_i)$ for $1 \le j \le \ell = \ell(\mu)$. For example, if $\mu = (3,3,2) \vdash 8$, then $\sigma_\mu = t_1 t_2 t_4 t_5 t_7 \in \tilde{S}_8$ and $\theta_8(\sigma_\mu) = (123)(456)(78) \in S_8$.

Since $\ker(\theta_n) = \{\pm 1\}$, every $\sigma \in \tilde{S}_n$ is conjugate to σ_μ or $-\sigma_\mu$ for some partition μ of n.

THEOREM 2.3. (Schur) *Let μ be a partition of n. Then the elements σ_μ and $-\sigma_\mu$ are not conjugate in $\tilde{S}_n$ iff either $\mu \in OP_n$ or $\mu \in DP_n^-$.*

Proof. See [St1] or [J]. □

Let $\Omega_{\mathbf{Q}} = \coprod_{n \ge 0} \Omega_{\mathbf{Q}}^n$ denote the graded subring of $\Lambda_{\mathbf{Q}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ generated by $1, p_1, p_3, \ldots$ and let $\Omega = \Lambda \cap \Omega_{\mathbf{Q}}$ denote the $\mathbf{Z}$-coefficient graded subring of $\Omega_{\mathbf{Q}}$. Clearly $\{p_\lambda \mid \lambda \in OP_n\}$ forms a basis of $\Omega_{\mathbf{Q}}^n$ and $\dim_{\mathbf{Q}} \Omega_{\mathbf{Q}}^n = |OP_n|$.

DEFINITION 2.4. Define an inner product $[\ ,\ ]$ on $\Omega_{\mathbf{C}}^n$ by setting

$$[p_\lambda, p_\mu] = z_\lambda 2^{-\ell(\lambda)} \delta_{\lambda\mu} \quad \text{for } \lambda, \mu \in OP_n,$$

where

$$\delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.5. (Mac)

(1) *$\{Q_\lambda \mid \lambda \in DP_n\}$ is a basis of $\Omega_{\mathbf{Q}}^n$.*

(2) *$[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$,*

where P_λ (resp., Q_λ) is the Hall-Littlewood symmetric P-function (resp., Q-function) corresponding to a partition $\lambda \in DP$.

We now describe the irreducible spin characters of $\tilde{S}_n$ using the Hall-Littlewood symmetric functions P_λ and Q_λ

THEOREM 2.6. (Schur) *Define a class function* φ^λ *for each* $\lambda \in DP_n^+$ *by*

$$\varphi^\lambda(\sigma_\mu) = \begin{cases} [2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\ 0 & \text{otherwise} \end{cases}$$

and define a pair of class functions $\varphi_\pm^\lambda$ *for each* $\lambda \in DP_n^-$ *via*

$$\varphi_\pm^\lambda(\sigma_\mu) = \begin{cases} \frac{1}{\sqrt{2}}[2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\ \pm i^{(n-\ell(\lambda)+1)/2}\sqrt{\frac{1}{2}z_\lambda} & \text{if } \mu = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ *if* $\lambda = 1^{m_1}2^{m_2}\cdots$.

Then the class functions $\varphi^\lambda(\lambda \in DP_n^+)$ *and* $\varphi_\pm^\lambda(\lambda \in DP_n^-)$ *are the irreducible spin characters of* $\tilde{S}_n$.

Proof. See [St1] or [J]. □

Although Theorem 2.6 determines the irreducible spin characters φ^λ, it is difficult to use Theorem 2.6 to evaluate $\varphi^\lambda(\sigma_\mu)$ explicitly for $\mu \in OP$. But in [Mo] Morris has derived a recurrence for the evaluation of these characters which is similar to the well-known Murnaghan-Nakayama formula for ordinary characters of S_n.

Recently Stembridge [St2] gave a combinatorial reformulation for Morris' recurrence using shifted tableaux, rather than the machinery of Hall-Littlewood functions used by Morris. We now describe Stembridge's interpretation for Morris' rule.

LEMMA 2.7. (Stembridge) *Let* k *be an odd number and* $|\lambda/\mu| = k$. *Then*

(1) $[Q_{\lambda/\mu}, p_k] = 0$ *unless* λ/μ *is a rim hook.*
(2) $[Q_{\lambda/\mu}, p_k] = (-1)^r$ *if* λ/μ *is a single rim hook of rank* r.
(3) $[Q_{\lambda/\mu}, p_k] = 2(-1)^r$ *if* λ/μ *is a double rim hook of rank* r.

Proof. See [St2]. □

THEOREM 2.8. (Stembridge) *For any* $\gamma \in OP$, *we have*

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_S w(S),$$

where the sum is over all shifted rim hook tableaux S of shape λ/μ and content γ.

Proof. Since the P_λ's and Q_λ's are dual bases, we have

$$p_r P_\mu = \sum_{\lambda \in DP} [p_r P_\mu, Q_\lambda] P_\lambda \quad \text{for any odd integer } r.$$

By iterating this expansion successively for $r = \gamma_1, \ldots, \gamma_\ell$, we find

$$[p_\gamma P_\mu, Q_\lambda] = \sum_{\{\lambda^j\}} [p_{\gamma_1} P_{\lambda^0}, Q_{\lambda^1}] \cdots [p_{\gamma_\ell} P_{\lambda^{\ell-1}}, Q_{\lambda^\ell}],$$

where $\mu = \lambda^0, \lambda = \lambda^\ell$. Since $[Q_{\lambda/\mu}, P_\nu] = [Q_\lambda, P_\mu P_\nu]$ and the P_ν's span $\Omega_{\mathbf{Q}}$, $[Q_{\lambda/\mu}, f] = [Q_\lambda, f P_\mu]$ for any $f \in \Omega_{\mathbf{Q}}$, and therefore

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_{\{\lambda^j\}} [Q_{\lambda^1/\lambda^0}, p_{\gamma_1}] \cdots [Q_{\lambda^\ell/\lambda^{\ell-1}}, p_{\gamma_\ell}].$$

Note that $Q_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$. Thus the only nonzero contributions to $[Q_{\lambda/\mu}, p_\gamma]$ in this expansion occur when $\lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^\ell$ and $|\lambda^i| - |\lambda^{i-1}| = \gamma_i$ $(1 \leq i \leq \ell)$. Hence it suffices to evaluate $[Q_{\lambda/\mu}, p_k]$ for all skew shapes λ/μ of weight k (k odd), and the description of $[Q_{\lambda/\mu}, p_k]$ in Lemma 2.7 gives a complete proof of Theorem 2.8. □

EXAMPLE 2.9. Consider $\lambda = (6,3,2,1)$, $\gamma = (5,3,3,1)$. There are four shifted rim hook tableaux of shape λ and content γ. See *Figure 2.1*. Since $w(T_1) = w(T_2) = w(T_3) = -2$ and $w(T_4) = 4$, $[Q_{6321}, p_1 p_3^2 p_5] = -2$. Therefore Theorem 2.6 implies that

$$\varphi^{6321}(\sigma_{5331}) = [Q_{6321}, p_1 p_3^2 p_5] = -2.$$

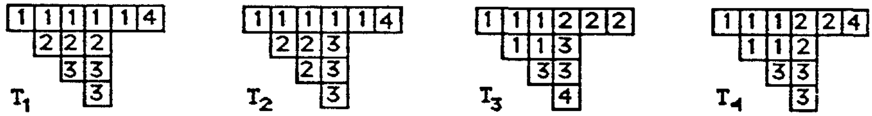

Figure 2.1

3. Swiching rule on the shifted rim hook tableaux.

In this section we prove the switching rule showing that the sum of the weights of the shifted rim hook tableaux of a given shape and content does not depend on the order of the content if content parts are all odd.

DEFINITION 3.1. P_2 is said to be a ***shifted second tail circled $*$-rim hook tableau*** if P_2 is a shifted second tail circled rim hook tableau whose entries include $*$ and are from the set $\{1, 2, \ldots, m, *\}$, where $r-1 < * < r$ for some integer r. We introduce the symbol $*$ to make it clear that no established order relationship governs $*$. We say that $*$ is covered by r (denoted by $* \lessdot r$) if r is the next integer larger than $*$ in P_2.

From now on, unless we explicitly specify to the contrary, we assume P_2 is a shifted second tail circled $*$-rim hook tableau of shape λ and contents all odd and $* \lessdot r$ in P_2. The circling of the second tail is necessary to compensate for the weight of 2 on double rim hooks.

DEFINITION 3.2. If $\kappa\langle * \rangle \cup \kappa\langle r \rangle$ is disconnected in P_2, we call $*$ and r ***disconnected.*** We say that $*$ and r is a ***single*** (resp., ***double)*** ***rim hook union*** if $\kappa\langle * \rangle \cup \kappa\langle r \rangle$ is a single (resp., double) rim hook. If $\kappa\langle * \rangle \cup \kappa\langle r \rangle$ is neither disconnected nor any rim hook union, we call $*$ and r ***overlapping.***

We define an assignment $X(*)$ that sends P_2 into another shifted second tail circled $*$-rim hook tableau $\hat{P}_2$ of shape λ as follows:

1. If $*$ and r are disconnected in P_2, then $X(*)P_2 = \hat{P}_2 = P_2$, but with $r \lessdot *$.

2. If $*$ and r is a single rim hook union, then $X(*)$ moves all of the symbols at the head of $\tau = \kappa\langle * \rangle \cup \kappa\langle r \rangle$ to the tail of τ, and vice versa. The number of r's and $*$'s is preserved. In this case, either $r \lessdot *$ in $\hat{P}_2$ or $* \lessdot r$ in $\hat{P}_2$. *Figure 3.1* gives us an example for case 2 with $* \lessdot r$ in $\hat{P}_2$ and *Figure 3.2* shows case 2 with $r \lessdot *$ in $\hat{P}_2$.

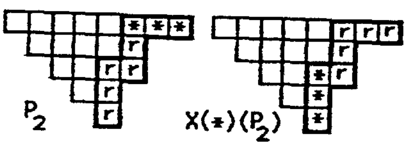

Figure 3.1

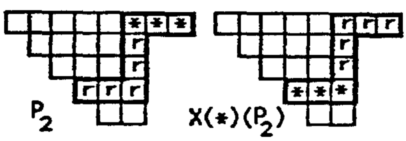

Figure 3.2

3. If $*$ and r is a double rim hook union, let $\tau = \kappa\langle * \rangle \cup \kappa\langle r \rangle$. Recall that we can write τ as follows: $\tau = \beta_1 \cup \beta_2 = \gamma_1 \cup \gamma_2 = \mathcal{A} \cup \alpha_1$.

Let $a = |\kappa\langle * \rangle|, b = |\kappa\langle r \rangle|$ and $c = |\beta_1| = |\gamma_1|$. Then we have

$$|\beta_2| = |\gamma_2| = a + b - c,$$
$$|\alpha_1| = 2c - a - b \quad \text{and}$$
$$|\mathcal{A}| = 2(a + b - c).$$

DEFINITION 3.3. We say we *fill* τ from β_1 if the word with a $*$'s followed by b r's is inserted in τ, starting at the head of β_1, running down β_1 to the diagonal, then up β_2. Similarly, define filling τ from β_2, from γ_1 and from γ_2.

It is not hard to verify the following two lemmas. For examples, see *Figure 3.3* and *Figure 3.4*.

LEMMA 3.4. *If $a, b \neq |\mathcal{A}|/2$ then there are exactly two shifted skew rim hook tableaux of shape τ with a $*$'s and b r's. One of these (say T_1) fills τ from β_1 or from γ_1. The other (say T_2) fills τ from β_2 or from γ_2. If $* < r$ in T_1 and T_2 or if $r < *$ in T_1 and T_2, then $w(T_1) = -w(T_2)$. Otherwise, $w(T_1) = w(T_2)$.*

LEMMA 3.5. *If $a = |\mathcal{A}|/2$ (resp., $b = |\mathcal{A}|/2$), then there are exactly three shifted skew rim hook tableaux of shape τ with a $*$'s and b r's. In one of these (say T_4), β_2 will contain the $*$'s (resp., r's). In the second (say T_5), γ_2 will contain the $*$'s (resp., r's). The third (say T_6) fills τ from β_1 or from γ_1 (resp., from β_2 or from γ_2). Also, $w(T_4) = -w(T_5)$ and if $* < r$ in T_6 then $w(T_6) = w(T_4) - w(T_5)$ (resp., $w(T_6) =$*

$w(T_5) - w(T_4)$) while if $r < *$ in T_6 then $w(T_6) = w(T_5) - w(T_4)$ (resp., $w(T_6) = w(T_4) - w(T_5)$).

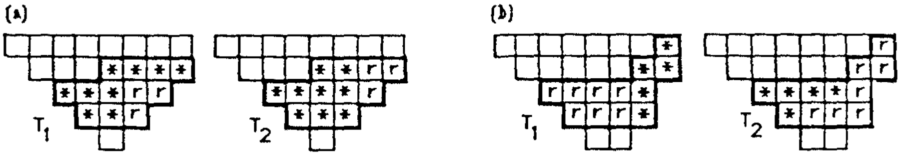

Figure 3.3

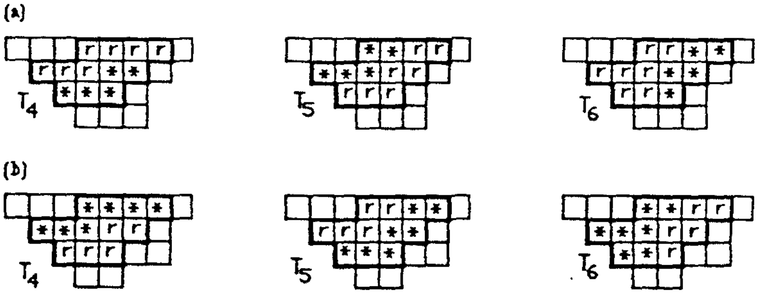

Figure 3.4

We now describe an assignment $X(*)P_2$ when $*$ and r is a double rim hook union in P_2. Suppose first $a, b \neq |\mathcal{A}|/2$. If P_2 contains T_1, then $X(*)P_2 = \hat{P}_2$ contains T_2, and vice versa. See *Figure 3.5.*

Suppose now $a = |\mathcal{A}|/2$ or $b = |\mathcal{A}|/2$. Say $a = |\mathcal{A}|/2$. Since $b = c$ and $* \lessdot r$ in P_2, P_2 cannot contain T_4. If P_2 contains T_5, then $\hat{P}_2$ contains T_6 with no circle on the second tail of τ; if P_2 contains T_6 with no circle on the second tail of τ, then $\hat{P}_2$ contains T_5; if P_2 contains T_6 with a circle on the second tail of τ, then $\hat{P}_2$ contains T_4. See *Figure 3.6.*

4. If $*$ and r is overlapping, then $X(*)$ exchanges $*$ and r along diagonals of P_2. See *Figure 3.7.*

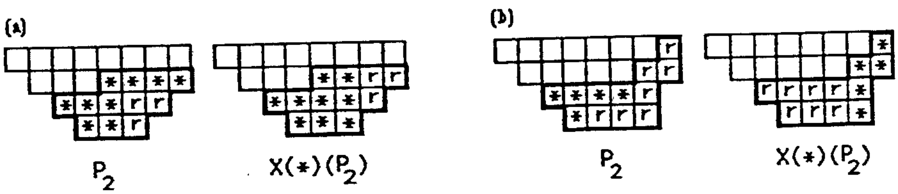

Figure 3.5

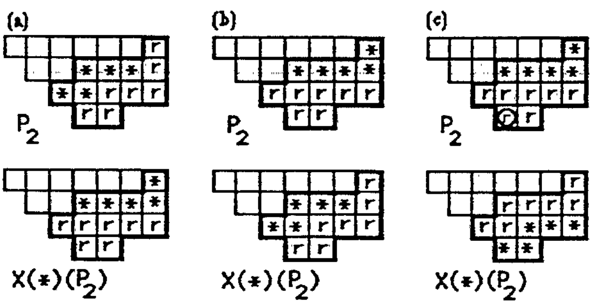

Figure 3.6

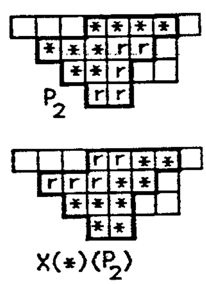

Figure 3.7

PROPOSITION 3.6. *We have $* < r$ in $\hat{P}_2 = X(*)P_2$ if and only if*

$$w'(\kappa_{P_2}\langle * \rangle)w'(\kappa_{P_2}\langle r \rangle) = -w'(\kappa_{\hat{P}_2}\langle * \rangle)w'(\kappa_{\hat{P}_2}\langle r \rangle),$$

and $r < *$ in $\hat{P}_2 = X(*)P_2$ if and only if

$$w'(\kappa_{P_2}\langle * \rangle)w'(\kappa_{P_2}\langle r \rangle) = w'(\kappa_{\hat{P}_2}\langle * \rangle)w'(\kappa_{\hat{P}_2}\langle r \rangle).$$

Proof. It is easy to verify the above statements with a case-by-case argument. □

From Proposition 3.6 we have the following theorem:

THEOREM 3.7. *Let λ be a partition with all distinct parts and $\rho \in OP_n$ and ρ' be any reordering of ρ. Then*

$$\sum_{P_2} w'(P_2) = \sum_{P'_2} w'(P'_2),$$

where the left-hand sum is over all shifted second circled rim hook tableaux P_2 of shape λ and content ρ, and the right-hand sum is over all shifted second circled rim hook tableaux P'_2 of shape λ and content ρ'.

Proof. If ρ and ρ' differ by an adjacent transposition, X defined above establishes this identity. The theorem follows because any reordering can be written as a sequence of adjacent transpositions. The "signed bijection" in the general case is given by the involution principle of Garsia and Milne [GM]. See [SW1] for details. □

Theorem 2.8 and Proposition 1.9 imply the following corollaries:

COROLLARY 3.8. *Let $\lambda \in DP_n$. Let ρ have all odd parts and ρ' be any reordering of ρ. Then*

$$\sum_{\substack{P \text{ shifted rim hook} \\ \text{tableaux of shape } \lambda \\ \text{and content } \rho}} w(P) = \sum_{\substack{P' \text{ shifted rim hook} \\ \text{tableaux of shape } \lambda \\ \text{and content } \rho'}} w(P').$$

COROLLARY 3.9. *Let $\lambda \in DP_n$ and $\rho \in OP_n$. Then*

$$\varphi^{\lambda}_{\rho} = \varphi^{\lambda}_{\rho'},$$

where ρ' is any reordering of ρ.

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