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ON REGULARITY AND APPROXIMATION FOR HAMILTON–JACOBI EQUATIONS

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1. Introduction

Discontinuities may form in the derivative with respect to x of the solution $u(x,t)$ of the Hamiton-Jacobi equation

$$\text{(H-J)} \qquad \begin{aligned} u_t + f(\nabla u) &= 0, \quad x \in R^n, \quad t > 0, \\ u(x,0) &= u_0(x), \quad x \in R^n, \end{aligned}$$

even though the flux f and the initial data u_0 are smooth and u_0 has compact support. The difficulties in defining the correct solutions of (H-J) were overcome by Crandall and Lions [2], who introduced the notion of viscosity solutions. They also showed that viscosity solutions of (H-J) are stable in L^∞ with respect to perturbation in the initial data. Consequently the space of Lipschitz continuous functions forms a regularity space for (H-J). In one space dimension, Hong [6] has recently shown that if f is strictly convex, has three bounded derivatives, and $u_0' \in BV(R)$, then for a finite interval I, $0 < \alpha < 3$ and $q \in (0,\infty]$, $A_q^\alpha(C(I))$ are also regularity spaces for (H-J). As a corollary, since the Besov spaces $B_q^{\alpha-1}(L^q)$ for $1 < \alpha < 3$ and $q = 1/\alpha$ are equivalent to $A_q^\alpha(C(I))$, it follows that whenever u_0' is in $B_q^{\alpha-1}(L^q)$, $u_x(\cdot,t)$ remains in the same space for all positive time. We will prove this fact directly.

It may be useful to sketch the work by Hong [6]. He first showed that solutions of (H-J) are stable under perturbations in the nonlinearity f as well as the initial data u_0. In particular, if $u(x,t)$ solves (H-J) and $v(x,t)$ solves the similar problem

$$v_t + g(\nabla v) = 0, \quad x \in R^n, \quad t > 0,$$

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with initial data $v(x,0) = v_0(x)$, then for positive t,

(H) $\|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(R^n)} \le \|u_0 - v_0\|_{L^\infty(R^n)} + t\|f - g\|_{L^\infty(R^n)}$.

A specific construction in one space dimension is then made in which f is approximated in $L^\infty(R)$ to order $(2^n)^{-3}$ by a C^1 piecewise quadratic function f_n and u_0 is approximated in L^∞ by a continuous, piecewise quadratic polynomial v_0 with 2^n free knots. It is then shown that the solution of $v(\cdot,t)$ of

$$v_t + f_n(v_x) = 0, \quad x \in R, \quad t > 0,$$
$$v(x,0) = v_0(x), \quad x \in R,$$

is again continuous, piecewise quadratic for all time and has no more than $C2^n$ pieces where for some C. The Hong's stability theorem (H) shows that $u(\cdot,t)$ can be approximated with an error not exceeding the error of approximation of u_0 plus $O((2^n)^{-3})$. The regularity theorem is then proved by using the characterization developed by DeVore and Popov [4], using the result of Petrushev [11].

This approach does not carry over directly to the spaces $B_q^{\alpha-1}(L^q)$ because one must deal with $u_x(\cdot,t)$. To overcome this difficulty, we use the relationship between the equations of (H-J) and hyperbolic conservation laws. This relationship is very simple: if u is the viscosity solution of (H-J), then $v = u_x$ is the entropy solution of scalar conservation laws

(C)
$$v_t + f(v)_x = 0, \quad x \in R, \quad t > 0,$$
$$v(x,0) = v_0(x) = u_0'(x), \quad x \in R.$$

Moreover, Lucier [9] showed a similar stability result like (H): if v solves (C) and w solves the similar problem

$$w_t + g(w)_x = 0, \quad x \in R, \quad t > 0,$$
$$w(x,0) = w_0(x), \quad x \in R,$$

then for positive t,

(L)
$$\|v(\cdot,t) - w(\cdot,t)\|_{L^1(R)} \le \|v_0 - w_0\|_{L^1(R)} + t\|f' - g'\|_{L^\infty(R)} \min\{\|u_0\|_{BV(R)}, \|w_0\|_{BV(R)}\}$$

Using the relationship between (H-J) and (C), and the stability result (L) given by Lucier, we prove the following theorem that is the main result of this paper.

THEOREM 1.1. *Let $\alpha \in (1,3)$. Suppose that u_0 is Lipschtz continuous, has support in $[0,1]$, and that $u_0' \in$ BV(R). Let $f(0) = 0$. Suppose that $f'' > 0$ and that f' and f''' are bounded in $\Omega = \{y||y| \leq C|u_0'|_{BV(R)}\}$, where C will be specified later. If u_0' is in $B_q^{\alpha-1}(L^q([0,1]))$, where $q = 1/\alpha$, then $u_x(\cdot,t)$ has support in $I_t = [t\inf_{\rho\in\Omega} f(\rho), 1 + t\sup_{\rho\in\Omega} f(\rho)]$ and $u_x(\cdot,t) \in B_q^{\alpha-1}(L^q(I_t))$.*

2. Preliminaries

Analysis by the method of characteristics shows that C^1 solutions of (C) are constant along the characteristic lines $x = x_0 + tf'(v_0(x_0))$, so near the line $t = 0$ the solutions of (C) are of the form

$$v(x,t) = v_0(x - f'(v)t). \tag{2.1}$$

Since discontinuities may develop in v, the implicit equation (2.1) may no longer be solution. However, the minimization property introduced by Lax [8] makes it possible for us to find local solutions of (2.1), at least when f is convex. He picks out a specific value $y := y(x,t)$ among many possible solutions of $\frac{x-y}{t} = f'(v_0(y))$ so that $v(x,t) = v_0(y)$. He also showed that $y(x,t)$ is an increasing function of x for fixed t. Moreover, if $y(x,t)$ is discontinuous at x, then shock occurs.

Let I be a finite interval. We are going to use the following notations

$$\|f\|_{L^p(I)}^* := \left(\frac{1}{|I|}\int_I |f(x)|^p\,dx\right)^{1/p},$$

if $0 < p < \infty$ and

$$\|f\|_{L^\infty(I)}^* := \sup_{x\in I} |f(x)|.$$

The following inequalities are well known for polynomials P of degree no more than k; see DeVore and Sharley [5].

For each $k = 0, 1, \ldots$ and p, $q \in (0,\infty]$ there is a constant C such that for all polynomials P of dgree $\leq k$,

$$\|P\|_{L^p(I)}^* \leq C\|P\|_{L^q(I)}^*. \tag{2.2}$$

For each $k = 0, 1, ...$ and $p \in (0, \infty]$ there is a constant C such that for all polynomials P of degree $\leq k$,

$$\|P'\|^*_{L^p(I)} \leq C|I|^{-1}\|P\|^*_{L^p(I)}. \tag{2.3}$$

Consider the functions in $L^1(I)$ and a finite interval I. For any $f \in L^1(I)$ and any positive integer n, let $E^2_n(f)_1 := inf\|f - \phi\|_{L^1(I)}$ where the infimum taken over all piecewise polynomial functions ϕ defined on I of degree less than 2 with at most $2^n - 1$ free interior knots (i.e., 2^n polynomial pieces). For $1 < \alpha < 3$ and $q = \frac{1}{\alpha}$, define $A^{\alpha-1}_q(L^1(I))$ to be the set of functions for which

$$\|f\|_{A^{\alpha-1}_q(L^1(I))} := \|f\|_{L^1(I)} + \Big(\sum_{n=0}^{\infty}[2^{n(\alpha-1)}E^2_n(f)_1]^q\Big)^{1/q} < \infty.$$

We now define Besov spaces. For $\alpha \in (0, \infty)$, $q \in (0, \infty]$ and $p \in (0, \infty]$, the Besov spaces $B^\alpha_q(L^p(I))$ are defined as follows: Pick any integer $r > \alpha$, let $\Delta^r_h f(x)$ be the forward difference of f at x with step size h (i.e., $\Delta^0_h f(x) := f(x)$ and $\Delta^r_h f(x); = \Delta^{r-1}_h f(x+h) - \Delta^{r-1}_h f(x)$). Let $I_h = \{x \in I \mid x + rh \in I\}$. Define $\omega_r(f, t)_p = \sup_{|h|<t} \|\Delta^r_h f\|_{L^p(I_h)}$. The Besov space $B^\alpha_q(L^p(I))$ is defined to be the set functions f for which

$$\|f\|_{B^\alpha_q(L^p(I))} := \|f\|_{L^p(I)} + \Big(\int_0^\infty \left[t^{-\alpha}\omega_r(f, t)_p\right]^q dt/t\Big)^{1/q} < \infty.$$

We are particularly interested in the spaces $B^{\alpha-1}(I) := B^{\alpha-1}_q(L^q(I)$, $1 < \alpha < 3$, where $q = 1/\alpha$. Based on work by Petrushev [11], DeVore and Popov [4] showed the following theorem.

THEOREM 2.1. *If $1 < \alpha < 3$ and $q = 1/\alpha$, then*

$$B^{\alpha-1}_q(L^q(I)) = A^{\alpha-1}_q(L^1(I)).$$

3. Regularity for Hamilton-Jacobi equations

In this section we prove Theorem 1.1. The proof is divided into several steps; we first construct a certain approximation to the solution $v(x,t)$ of (C).

The initial data $v_0 = u_0'$ has support in $I = [0,1]$ since we assume that u_0 has support in I. We now fix $r = 2$. Assume that P_n is the best $L^1(I)$ approximation to v_0 having support in I if $\|v_0 - P_n\|_{L^1(I)} = E_n^2(v_0)_1$. Let $\{\tau_i\}_{i=0}^{2^n}$, with $\tau_0 = 0$ and $\tau_{2^n} = 1$, be the ordered set of knots P_n. Define each interval I_i by $I_i := [\tau_i, \tau_{i+1}]$ with $|I_i| := \tau_{i+1} - \tau_i$.

LEMMA 3.1. *If P_n is the best $L^1(I)$ approximation to v_0, then*

$$|P_n|_{BV(R)} \le |v_0|_{BV(R)} = |u_0'|_{BV(R)}.$$

proof. Assume that v_0 is right continuous. For each i let C_i be the number satisfying $C_i := v_0(x_i)$ for some $x_i \in I_i$. Suppose that $C(x)$ is the piecewise constant function taking the value C_i on each I_i. Then $|C(x)|_{BV(R)} \le |v_0|_{BV(R)}$. Therefore

$$\begin{aligned}
|P_n|_{BV(I_i)} &= \int_{I_i} |P_n'|\,dx \\
&= \int_{I_i} |P_n' - C_i'|\,dx \\
&\le \frac{C}{|I_i|}\int_{I_i} |P_n - C_i|\,dx \\
&\le \frac{C}{|I_i|}\int_{I_i} \{|P_n - v_0| + |v_0 - C_i|\}\,dx \qquad (3.1)\\
&\le \frac{C}{|I_i|}\int_{I_i} |v_0 - C_i|\,dx \\
&\le \frac{C}{|I_i|}\sup_{x\in I_i}\int_{\tau_i}^{x} |x-s||v_0'(s)|\,ds \\
&\le C\int_{I_i} |v_0'|\,dx \\
&\le C|v_0|_{BV(I_i)}.
\end{aligned}$$

Here the first inequality is (2.3).

We now measure the jump $|P_n(\tau_i{}^+) - P_n(\tau_i{}^-)|$. It is clear that

$$|P_n(\tau_i{}^+) - P_n(\tau_i{}^-)| \le \|P_n(x) - C_i\|_{L^\infty(I_i)} + \|P_n(x) - C_{i-1}\|_{L^\infty(I_{i-1})} + |C_i - C_{i+1}|,$$

and

$$\begin{aligned}\|P_n(x) - C_i\|_{L^\infty(I_i)} \le & \frac{C}{|I_i|}\int_{I_i} |P_n(x) - C_i|\, dx \\ \le & \frac{C}{|I_i|}\int_{I_i} |v_0 - C_i|\, dx \\ \le & C|v_0|_{BV(I_i)}.\end{aligned}$$

The second conclusion is (3.1). Hence the jump is not larger than

$$C|v_0|_{BV(I_i)} + C|v_0|_{BV(I_{i+1})} + |C_{i+1} - C_i|.$$

So we get

$$\begin{aligned}|P_n|_{BV(R)} &\le C\big(|v_0|_{BV(R)} + |C(x)|_{BV(R)}\big) \\ &\le C|v_0|_{BV(R)}.\end{aligned}$$

This completes the proof. □

This C is the constant of Theorem 1.1. Moreover one can easily see that

$$\|P_n\|_{L^\infty(R)} \le \|v_0\|_{L^\infty(R)} = \|u_0'\|_{L^\infty(R)}.$$

So, the ranges of P_n and v_0 are contained in $\Omega = \{y \mid |y| \le C|u_0'|_{BV(R)}\}$.

We now construct an approximation f_n to f on Ω. There is a C^1 piecewise quadratic approximation f_n to f with knots at the points $j/2^n$, $j \in Z$, that is defined by $f_n'(j/2^n) = f'(j/2^n)$ with $f_n'(x)$ is a linear function between $j/2^n$ and $j+1/2^n$ and $f_n(0) = f(0)$. Moreover $\|f_n' - f'\|_{L^\infty(R)} \le C\|f'''\|_{L^\infty(R)}\frac{1}{(2^n)^2}$; see [1] and [3]. Consider now the perturbed problem

$$\begin{aligned}(P_n)_t + f_n(P_n)_x = 0, &\quad x \in R, \quad t > 0,\\ P_n(x,0) = P_n(x), &\quad x \in R.\end{aligned}$$

It is shown in [9] and [10] that the solution $P_n(\cdot,t)$ is again piecewise linear for each time $t > 0$ and the number of knots is no more than $C2^n$ where C depends only on t, $|v_0|_{BV(R)}$ and the degree 1. The stability theorem (L) then shows that

$$\begin{aligned}
&\|v(\cdot,t) - P_n(\cdot,t)\|_{L^1(R)} \\
&\quad \le \|v_0 - P_n(\cdot,0)\|_{L^1(R)} + Ct\|f' - f_n'\|_{L^\infty(R)}|v_0|_{BV(R)} \\
&\quad \le \|v_0 - P_n(\cdot,0)\|_{L^1(R)} + Ct\|f'''\|_{L^\infty(R)}\frac{1}{(2^n)^2}.
\end{aligned} \tag{3.2}$$

Proof of Theorem 1.1. The first conclusion is classical. We will prove the second conclusion. We obtain from (3.2) that $P_n(\cdot,t) \to v(\cdot,t)$ in $L^1(I_t)$ and so we write

$$u(\cdot,t) = \sum_{n=-1}^{\infty} (P_{n+1}(\cdot,t) - P_n(\cdot,t)),$$

where $P_{-1}(\cdot,t) = 0$. We now count some points with respect to x. The first are the points of intersection of $P_{n+1}(x,t)$ and $P_n(x,t)$ and the second type of points are points where either $P_{n+1}(x,t)$ or $P_n(x,t)$ is discontinuous in x. The total number of points of these two types is clearly no more than $C2^n$. One can therefore see that $P_{n+1}(x,t) - P_n(x,t)$ is a discontinuous, piecewise linear function in x with at most $C2^n$ pieces. Let each interval I_i be determined by two adjacent points from type one and type two. Let $P_{n+1}(x,t) - P_n(x,t) = \sum_{i=1}^{K} L_i$, $K \le C2^n$, where each linear polynomial piece L_i, in x at fixed time t, is defined on I_i and vanishes outside I_i.

We will use a generic constant C depending on the constant C in (3.2) and $\|f^{(3)}\|_{L^\infty(R)}$. We fix h. For each i define three subsets A_i, B_i and C_i of I_i by $A_i = \{x \in I_i \mid x, x+h, x+2h \in I_i\}$, $B_i = \{x \in I_i \mid x \notin A_i \text{ and } \{x, x+h, x+2h\} \cap I_i \neq \phi\}$ and $C_i = I_i - (A_i \cup B_i)$. Then $A_i = \phi$ if $h > \frac{|I_i|}{2}$ and B_i has measure no more than $4\min(h, |I_i|)$.

For $x \in B_i$, we fix $p > 1$ with $p < \frac{4}{3}$. Since

$$\begin{aligned}
|\Delta_h^2(L_i,x)| &\le |L_i(x+2h) - 2L_i(x+h) + L_i(x)| \\
&\le 2|L_i(x+2h) + L_i(x+h) + L_i(x)| \\
&\le C|L_i(x)|,
\end{aligned}$$

by Hölder inequality, we have

$$\begin{aligned}
\int_{B_i} |\Delta_h^2(L_i,x)|^q\,dx \le& |B_i|^{1-q/p}\left(\int_{I_i} |L_i(x)|^p\,dx\right)^{q/p}\\
\le& C[\min(h,|I_i|)]^{1-q/p}\left(\int_{I_i} |L_i(x)|^p\,dx\right)^{q/p}\\
=& C[\min(h,|I_i|)]^{1-q/p}|I_i|^{q/p}\left(\|L_i\|^*_{L^p(I_i)}\right)^q\\
\le& C[\min(h,|I_i|)]^{1-q/p}|I_i|^{q/p}\left(\|L_i\|^*_{L^1(I_i)}\right)^q\\
\le& C[\min(h,|I_i|)]^{1-q/p}|I_i|^{q/p}\left(\|L_i\|^*_{L^1(I_i)}+\frac{1}{(2^n)^2}\right)^q\\
=& C[\min(h,|I_i|)]^{1-q/p}|I_i|^{-q+q/p}\left(\int_{I_i} |L_i(x)|\,dx+\frac{|I_i|}{(2^n)^2}\right)^q. \qquad (3.3)
\end{aligned}$$

Let $x \in C_i$. Since $\Delta_h^2(L_i,x)=0$,

$$\int_{C_i} |\Delta_h^2(L_i,x)|^q\,dx = 0. \qquad (3.4)$$

We finally consider $x \in A_i$. For each $x \in I_i$ there is $\xi \in I_i$ such that

$$|\Delta_h^2(L_i,x)| = Ch|L_i'(\xi)|,$$

because L_i is linear on I_i. Hence,

$$\begin{aligned}
\int_{A_i} |\Delta_h^2(L_i,x)|^q\,dx \le& Ch^q\int_{I_i} |L_i'(x)|^q\,dx\\
\le& Ch^q|I_i|^{-q}\int_{I_i} |L_i(x)|^q\,dx\\
\le& Ch^q|I_i|^{1-q}(\|L_i\|^*_{L^1(I_i)})^q\\
\le& Ch^q|I_i|^{1-2q}\left(\int_{I_i} |L_i(x)|\,dx+\frac{|I_i|}{(2^n)^2}\right)^q.
\end{aligned} \qquad (3.5)$$

Here the second inequality is (2.3) and the third inequality is (2.2).

(3.3), (3.4) and (3.5) now yield that

$$\int_R |\Delta_h^2(L_i,x)|^q\,dx \le C\Big(h^q|I_i|^{1-2q}\chi(h) + [\min(h,|I_i|)]^{1-q/p}|I_i|^{q/p-q}\Big) \tag{3.6}$$
$$\times\left(\int_{I_i}|L_i(x)|\,dx + \frac{|I_i|}{(2^n)^2}\right)^q,$$

where $\chi(h)$ is the characteristic function on $[0,|I_i|/h]$.

Because one can easily show that $\omega_2(L_i,h)_q^q$ is less than the right hand side of the inequality (3.6),

(3.7)

$$\begin{aligned}
&\int_0^\infty [h^{-(\alpha-1)}\omega_2(L_i,h)_q]^q\,dh/h \\
=&\int_0^\infty h^{-q(\alpha-1)}\omega_2(L_i,h)_q^q\,dh/h \\
\le& C\left(\int_{I_i}|L_i(x)|\,dx + \frac{|I_i|}{(2^n)^2}\right)^q \times \left(\int_0^\infty h^{-q(\alpha-1)-1}h^q|I_i|^{1-2q}\chi(h)\,dh\right. \\
&\left. + \int_0^\infty h^{-q(\alpha-1)-1}[\min(h,|I_i|)]^{1-q/p}|I_i|^{-q+q/p}\,dh\right) \\
\le& C\left(\int_{|I_i|}|L_i(x)|\,dx + \frac{|I_i|}{(2^n)^2}\right)^q\left(|I_i|^{1-2q}\int_0^{|I_i|} h^{-q(\alpha-2)-1}\,dh\right. \\
&\left. + |I_i|^{1-q}\int_{|I_i|}^\infty h^{-q(\alpha-1)-1}\,dh + |I_i|^{-q+q/p}\int_0^{|I_i|} h^{-q(\alpha-1)-1}h^{1-q/p}\,dh\right) \\
\le& C|I_i|^{1-q\alpha}\left(\int_{I_i}|L_i(x)|\,dx + \frac{|I_i|}{(2^n)^2}\right)^q \\
=& C\left(\int_{I_i}|L_i(x)|\,dx + \frac{|I_i|}{(2^n)^2}\right)^q,
\end{aligned}$$

because $1-q\alpha=0$.

Now, since $q<1$, we have

$$\omega_2(P_{n+1}-P_n,h)_q^q \le \sum_{i=1}^K \omega_2(L_i,h)_q^q. \tag{3.8}$$

It follows immediately that

$$
\begin{aligned}
&\int_0^\infty h^{-(\alpha-1)q}\omega_2(P_{n+1}-P_n,h)_q^q\,dh/h \\
&\le \sum_{i=1}^K \int_0^\infty h^{-(\alpha-1)q}\omega_2(L_i,h)_q^q\,dh/h \\
&\le C\sum_{i=1}^K\left(\int_{I_i}|L_i(x)|\,dx+\frac{|I_i|}{(2^n)^2}\right)^q \\
&\le CK^{1-q}\left(\|P_{n+1}(\cdot,t)-P_n(\cdot,t)\|_{L^1(I_t)}+\frac{|I_t|}{(2^n)^2}\right)^q \\
&\le CK^{q(\alpha-1)}\left(\|P_{n+1}(\cdot,t)-P_n(\cdot,t)\|_{L^1(I_t)}^q+\frac{|I_t|^q}{(2^n)^{2q}}\right) \\
&\le C2^{q(\alpha-1)n}\left(\|P_{n+1}(\cdot,t)-P_n(\cdot,t)\|_{L^1(I_t)}^q+\frac{|I_t|^q}{(2^n)^{2q}}\right).
\end{aligned}
\tag{3.9}
$$

Here the first inequality is (3.8), the second is (3.7), the third is Hölder inequality and the fourth follows from the facts that $1-q=q(\alpha-1)$ and $0<q<1$.

Using (3.8), (3.9) and Theorem 2.1, we have

$$
\begin{aligned}
&\int_0^\infty [h^{-(q-1)}\omega_2(u_x,h)_q]^q\,dh/h \\
&=\int_0^\infty [h^{-(q-1)}\omega_2(v,h)_q]^q\,dh/h \\
&\le \sum_{n=-1}^\infty \int_0^\infty [h^{-(q-1)}\omega_2(P_{n+1}-P_n,h)_q]^q\,dh/h \\
&\le C\sum_{n=-1}^\infty 2^{nq(\alpha-1)}\left(\|P_{n+1}(\cdot,t)-P_n(\cdot,t)\|_{L^1(I_t)}^q+2^{-2nq}\right) \\
&\le C\sum_{n=-1}^\infty 2^{nq(\alpha-1)}\left(\|P_n(\cdot,0)-v_0(\cdot,0)\|_{L^1(I)}^q+2^{-n2q}+2^{-2nq}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{i=-1}^{\infty} 2^{nq(\alpha-1)}\Big(E_n^2(v_0)_1^q + 2^{-2nq}\Big) \\
(3.10) \qquad &= C \sum_{i=-1}^{\infty} 2^{nq(\alpha-1)}\Big(E_n^2(u_0')_1^q + 2^{-2nq}\Big) \\
&\leq C\|u_0'\|^q_{B_q^{\alpha-1}(L^q([0,1]))} + C,
\end{aligned}
$$

because from (3.2), for $n = -1, 0, ...,$

$$
\begin{aligned}
&\|P_{n+1}(\cdot,t) - P_n(\cdot,t)\|^q_{L^1(I_t)} \\
&\leq \Big(\|P_{n+1}(\cdot,t) - v_0(\cdot,t)\|_{L^1(I_t)} + \|v_0(\cdot,t) - P_n(\cdot,t)\|_{L^1(I_t)}\Big)^q \\
&\leq \Big(\|P_{n+1}(\cdot,0) - v_0(\cdot,0)\|_{L^1(I)} + \|P_n(\cdot,0) - v_0(\cdot,0)\|_{L^1(I)} + \frac{1}{(2^n)^2}\Big)^q \\
&\leq C\Big(\|P_n(\cdot,0) - v_0(\cdot,0)\|^q_{L^1(I)} + 2^{-2nq}\Big)
\end{aligned}
$$

By (3.10),

$$
\|u(\cdot,t)\|_{B_q^{\alpha-1}(L^q(I_t))} \leq C\|u_0'\|_{B_q^{\alpha-1}(L^q(I))} + C.
$$

This completes the proof. □

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