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CONJUGATES OF (H_2)–EQUIVARIANT MAPS OF SYMMETRIC DOMAINS

MIN HO LEE

1. Introduction

Let $\mathbf{G}$ be a semisimple algebraic $\mathbf{Q}$-group, and let K be a maximal compact subgroup of the semisimple Lie group $G = \mathbf{G}(\mathbf{R})$. We assume that the symmetric space $D = G/K$ has a G-invariant complex structure. Let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be a torsion-free arithmetic subgroup $\mathbf{G}$, and let $X = \Gamma \backslash D$ be the corresponding arithmetic variety. We consider another semisimple algebraic Q-group $\mathbf{G}'$ and an arithmetic subgroup Γ' of $\mathbf{G}'$. As in the case of $\mathbf{G}$, we associate G', K', D' and $X' = \Gamma' \backslash D'$ to $\mathbf{G}'$. Let $\rho : G \to G'$ be a homomorphism of Lie groups with $\rho(\Gamma) \subset \Gamma'$, $\phi : X \to X'$ a morphism of varieties, and $\tau : D \to D'$ a holomorphic lifting of ϕ such that ρ and τ are equivariant,

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all} \quad g \in G \quad \text{and} \quad y \in D.$$

If $\mathbf{G}'$ is a symplectic group, the equivariant pair (ρ, τ) determines a Kuga fiber variety $\pi : Y \to X$ over the arithmetic variety X whose fibers are polarized abelian varieties (see e.g. [9], [4], [16, Chapter 4]). Various number-theoretic and geometric aspects of Kuga fiber varieties have been investigated over the years (see e.g. [1],[2],[4],[13] for some recent ones).

Motivated by the problem of the construction of Kuga fiber varieties, I. Satake investigated the problem of finding all equivariant pairs that satisfy, what he called, the (H_1)-condition(see [14],[15],[16]). In his investigation he reduced the problem to the one of finding equivariant pairs that satisfy a stronger condition called (H_2)-condition. Kuga fiber varieties associated to (H_2)-equivariant pairs are known to be rigid (see e.g. [3, Proposition 1.2.1], and some results concerning these varieties

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were obtained recently. In [1] S. Abdulali investigated zeta functions of Kuga fiber varieties associated to (H_2)-equivariant pairs. He also obtained some results about the Hodge cycles([2]) and bottom fields ([3], see also [13]) of Kuga fiber varieties associated to (H_2)- equivariant pairs.

As is described above, the base space X of a Kuga fiber variety $\pi : Y \to X$ is an arithmetic variety. If $\sigma \in Aut(\mathbf{C})$, then it is well-known ([7],[8],[6],[12]) that the conjugate X^σ of the arithmetic variety X is also an arithmetic variety. In [10] the conjugates of Kuga fiber varieties themselves were treated. In that paper, the conjugate of the equivariant pair (ρ, τ) was constructed and it was proved that the conjugate $\pi^\sigma : Y^\sigma \to X^\sigma$ of the Kuga fiber variety $\pi : Y \to X$ is a Kuga fiber variety associated to this conjugate equivariant pair.

The purpose of this paper is to prove that if (ρ, τ) is an (H_2)-equivariant pair then the conjugate of (ρ, τ) constructed in [10] is also an (H_2)-equivariant pair. It follows from this result that the conjugate of a Kuga fiber variety associated to an (H_2)-equivariant pair is also a Kuga fiber variety associated to an (H_2)-equivariant pair.

1. The equivariance of conjugate pairs

In this section we shall describe some of the results which are contained in [10]. Let $\mathbf{G}$ be a simply connected semisimple algebraic group defined over $\mathbf{Q}$ that does not contain direct factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$. Let $G = \mathbf{G}(\mathbf{R})$ be the group of real points of $\mathbf{G}$ and let K be a maximal compact subgroup of G. We assume that the symmetric space $D = G/K$ has a G-invariant complex structure.

Let Γ be a torsion free arithmetic subgroup of $\mathbf{G}$ and let $X = \Gamma \setminus D$ be the corresponding arithmetic variety. Let $\{\Gamma_k \mid k = 1, 2, 3, \dots\}$ be a cofinal system of subgroups of finite index of Γ such that each Γ_k is an arithmetic subgroup and

$$\Gamma_k \subset \Gamma_j \quad \text{for} \quad j < k.$$

Then each quotient $X_k = \Gamma_k \setminus D$ is an arithmetic variety, and the collection $\{X_k \mid k \geq 1\}$ is a projective system of finite unramified covering manifolds of X. The projective limit

$$\hat{D} = \varprojlim X_k$$

has a natural structure of a non-connected complex manifold, which does not depend on the representation of $\hat{D}$ as a projective limit.

Let $\pi : D \to X$ and $\pi_k : X_k \to X$ be the canonical projections. To construct a mapping $\hat{\mu} : D \to \hat{D}$, we fix a point $x \in X$ and two elements $d_0 \in D$ and $\hat{d}_0 \in \hat{D}$ such that

$$\pi(d_0) = \hat{\pi}(\hat{d}_0) = x.$$

Then for each k there is a unique map $\mu_k : D \to X_k$ such that $\pi_k \circ \mu_k = \pi$. We define the embedding $\hat{\mu} : D \to \hat{D}$ by

$$\hat{\mu} = \varprojlim \mu_k.$$

PROPOSION 1.1. *$\hat{\mu}(D)$ is a connected component of $\hat{D}$ and is dense in $\hat{D}$.*

Proof. See [10, Proposition 1.1].

Let $\Gamma(X) \subset Aut(D)$ be the fundamental group of X and let $G_a(X)$ be the commensurability group of $\Gamma(X)$ in $Aut(D)$, i.e.,

$$G_a(X) = \{g \in Aut(D) \mid [\Gamma(X) : g\Gamma(X)g^{-1} \cap \Gamma(X)] < \infty$$

$$\text{and} \quad [\Gamma(X) : g^{-1}\Gamma(X)g \cap \Gamma(X)] < \infty\}.$$

The natural homomorphism $G \to Aut(D)$ induces the homomorphism

$$\alpha : \mathbf{G}(\mathbf{Q}) \longrightarrow Aut(D).$$

Since $\mathbf{G}$ has no factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$, the kernel of α is the center Z_Q of $\mathbf{G}(\mathbf{Q})$ and therefore it is finite. The image of $\Gamma \subset \mathbf{G}(\mathbf{Q})$ under α coincides with $\Gamma(X)$. If $\hat{\pi} : \hat{D} \to X$ denotes the natural projection, we set

$$\hat{G}_a(X) = Aut(\hat{D}), \quad \hat{\Gamma}(X) = \{\hat{g} \in \hat{G}_a(X) \mid \hat{\pi} \circ \hat{g} = \hat{\pi}\}.$$

Then $\hat{G}_a(X)$ is a complete locally compact topological group relative to the topology in which a basis of neighborhoods of the identity consists of the subgroups of finite index in $\hat{\Gamma}(X)$. To define a homomorphism $\hat{\chi} : G_a(X) \to \hat{G}_a(X)$, we take an element $g \in G_a(X)$. For each k, g induces a map $\hat{\chi}_k(g) : \hat{D} \to X_k$ (see [7, p.158]). We set

$$\hat{\chi}(g) = \varprojlim \hat{\chi}_k(g) \in Aut(\hat{D})$$

Then we have

$$\hat{\mu}(gd) = \hat{\chi}(g)\hat{\mu}(d) \quad \text{for} \quad d \in D \quad \text{and} \quad g \in G_a(X).$$

Since $\hat{\mu}(D)$ is dense in $\hat{D}$, we have

$$\hat{\chi}(g_1 g_2) = \hat{\chi}(g_1)\hat{\chi}(g_2) \quad \text{for} \quad g_1,\ g_2 \in G_a(X);$$

hence $\hat{\chi} : G_a(X) \to \hat{G}_a(X)$ is a homomorphism.

PROPOSITION 1.2. *(i) Let $\hat{d}_0$ be an element of $\hat{\mu}(D)$. Then $\hat{\chi}$ is an isomorphism between $G_a(X)$ and the subgroup of $\hat{G}_a(X)$ consisting of all $\hat{g} \in \hat{G}_a(X)$ such that $\hat{d}_0\hat{g} \in \hat{\mu}(D)$.*
(ii) $G_a(X)$ is dense in $\hat{G}_a(X)$.

Proof. See [6, Lemma 4 and Theorem 1].

We fix an element $\sigma \in Aut(\mathbf{C})$, and consider the complex variety X^σ obtained from X by the base change. Let D^σ be the universal covering manifold of X^σ, and let

$$\Gamma^\sigma = \Gamma(X^\sigma) \subset Aut(D^\sigma)$$

be the fundamental group of X^σ. If the varieties X_k^σ are the conjugates of X_k, we set

$$\hat{D}^\sigma = \varprojlim X_k^\sigma, \quad \hat{G}_a(X^\sigma) = Aut(\hat{D}^\sigma).$$

Then $\hat{G}_a(X^\sigma)$ is a complete locally compact topological group in the topology of subgroups of finite index in

$$\hat{\Gamma}(X^\sigma) = \{\hat{g}^\sigma \in \hat{G}_a(X^\sigma) \mid \hat{\pi}^\sigma \circ \hat{g}^\sigma = \hat{\pi}^\sigma\},$$

where $\hat{\pi}^\sigma : \hat{D}^\sigma \to X^\sigma$ is the natural projection. As in the case of X, we can construct the homomorphism $\hat{\chi}^\sigma : G_a(X^\sigma) \to \hat{G}_a(X^\sigma)$.

Let $V \subset G_a(X)$ be a subgroup with $[\Gamma(X) : V \cap \Gamma(X)] < \infty$, and let $\hat{V}$ be the closure of $\hat{\chi}(V)$ in $\hat{G}_a(X)$. If $\hat{g} \in \hat{V} \subset \hat{G}_a(X)$, then there are morphisms $g_k : Z_k \to X_k$ such that

$$\varprojlim Z_k = \hat{D} = \varprojlim X_k, \quad \hat{g} = \varprojlim g_k.$$

Applying σ to g_k, we obtain morphisms $g_k^\sigma : Z_k^\sigma \to X_k^\sigma$. Then

$$\hat{g}^\sigma = \varprojlim g_k^\sigma$$

is an element of $\hat{G}_a(X^\sigma) = Aut(\hat{D}^\sigma)$. We set $\hat{V}^\sigma = \{\hat{g}^\sigma \mid \hat{g} \in \hat{V}\}$ and define the subgroup V^σ of $G_a(X^\sigma)$ by

$$V^\sigma = (\hat{\chi}^\sigma)^{-1}(\hat{V}^\sigma \cap \mathrm{Im}(\hat{\chi}^\sigma)).$$

Now let G_a be a subgroup of finite index in $\alpha(\mathbf{G}(\mathbf{Q}))$ containing $\alpha(\Gamma) = \Gamma(X)$, and let

$$G_a^\sigma = (G_a)^\sigma \subset G_a(X^\sigma).$$

PROPOSITION 1.3. *G_a^σ is dense in the connected component of the identity of $Aut(D^\sigma)$ in the ordinary topology.*

Proof. G_a^σ is contained in the connected component of the identity of $Aut(D^\sigma)$ by [12, Lemma 3.7]. The density follows from [7, Theorem 5](see also [7, Theorem A.7]).

THEOREM 1.4. *The group $\Gamma^\sigma = \Gamma(X^\sigma)$ is an arithmetic subgroup of the connected component of the identity of $Aut(D^\sigma)$.*

Proof. This follows from the main theorems in [7] and [8].

Let $\mathbf{G}'$ be another semisimple algebraic $\mathbf{Q}$-group and let $\Gamma' \subset \mathbf{G}'(\mathbf{Q})$ be a torsion-free arithmetic subgroup. As in the case of $\mathbf{G}$, we associate G', K', D' and the arithmetic variety $X' = \Gamma' \setminus D'$ to $\mathbf{G}'$. Let $\rho : G \to G'$ be a homomorphism with $\rho(\Gamma) \subset \Gamma'$, $\phi : X \to X'$ a morphism of varieties, and $\tau : D \to D'$ a holomorphic lifting of ϕ such that ρ and τ are equivariant, i.e.,

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all} \quad g \in G \quad \text{and} \quad y \in D.$$

Let $\{\Gamma_k\}, \{X_k\}$ and $\hat{D}$ be as in §1, and let $\{\Gamma'_k\}$ be a cofinal system of arithmetic subgroups of finite index of Γ' such that $\rho(\Gamma_k) \subset \Gamma'_k$ for each $k \geq 1$. The quotient spaces $X'_k = \Gamma'_k \setminus D'$ are arithmetic varieties and they form a projective system $\{X'_k\}$ of covering manifolds of X'. The holomorphic map $\tau : D \to D'$ induces a morphism $\phi : X_k \to X'_k$ for each $k \geq 1$. We set

$$\hat{D}' = \varprojlim X'_k,$$

$$\hat{G}_a(X') = Aut(\hat{D}'),$$

and define the holomorphic map $\hat{\tau} : \hat{D} \to \hat{D}'$ by

$$\hat{\tau} = \varprojlim \phi_k.$$

The homomorphism ρ induces a homorphism $\hat{\rho} : \hat{G}_a \to \hat{G}_a(X')$ such that

$$\hat{\tau}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\tau}(\hat{y}) \quad \text{for all} \quad \hat{g} \in \hat{G} \quad \text{and} \quad \hat{y} \in \hat{D}$$

(see [10, Proposition 2.1]).

We fix an element $\sigma \in Aut(\mathbf{C})$. The varieties $X^\sigma, X'^\sigma, X_k^\sigma$ and $X_k'^\sigma$ are arithmetic varieties, and the collections $\{X_k^\sigma\}$ and $\{X_k'^\sigma\}$ are projective systems of finite unramified covering manifolds of X^σ and X'^σ respectively. Let $\phi_k^\sigma : X_k^\sigma \to X_k'^\sigma$ be the conjugate morphism of $\phi_k : X_k \to X'_k$ for each k. Let $\hat{D}, : \hat{G}_a(X^\sigma)$ and $\hat{\Gamma}(X^\sigma)$ be as in §1, and let

$$\hat{D}'^\sigma = \varprojlim X_k'^\sigma, \quad \hat{G}_a(X'^\sigma) = Aut(\hat{D}'^\sigma),$$

$$\hat{\Gamma}(X'^\sigma) = \{\hat{g}'^\sigma \in \hat{G}_a(X'^\sigma) \mid \hat{\pi}'^\sigma \circ \hat{g}'^\sigma = \hat{\pi}'^\sigma\},$$

where $\hat{\pi}'^\sigma : \hat{D}'^\sigma \to X'^\sigma$ is the natural projection. We define the map $\hat{\tau}^\sigma : \hat{D}^\sigma \to \hat{D}'^\sigma$ by

$$\hat{\tau}^\sigma = \varprojlim \phi_k^\sigma.$$

Then $\hat{\rho}$ induces a homorphism $\hat{\rho}^\sigma : \hat{G}_a^\sigma \to \hat{G}_a(X'^\sigma)$ such that $\hat{\rho}(\hat{\Gamma}(X^\sigma)) \subset \hat{\Gamma}(X'^\sigma)$ and

$$\hat{\tau}^\sigma(\hat{g}^\sigma \hat{y}^\sigma) = \hat{\rho}^\sigma(\hat{g}^\sigma)\hat{\tau}^\sigma(\hat{y}^\sigma)$$

for all $\hat{g}^\sigma \in \hat{G}^\sigma$ and $\hat{y}^\sigma \in \hat{D}^\sigma$. By Proposition 1.2 we can identify G_a^σ and $G_a(X'^\sigma)$ as subgroups of $\hat{G}_a^\sigma$ and $\hat{G}_a(X'^\sigma)$ respectively.

PROPOSITION 1.5. *If D'^σ is the connected component of $\hat{D}'^\sigma$ chosen as above, then $\hat{\rho}^\sigma(G_a^\sigma)$ is contained in $G_a(X'^\sigma)$.*

Proof. See [10, Proposition 4.1].

Now we set

$$\rho^\sigma = \hat{\rho}^\sigma\,|_{G_a^\sigma}, \quad \tau^\sigma = \hat{\tau}^\sigma\,|_{D^\sigma}\ .$$

Then we obtain a homomorphism $\rho^\sigma : G_a^\sigma \to G_a(X'^\sigma)$ and a holomorphic map $\tau^\sigma : D^\sigma \to D'^\sigma$ satisfying the relation

$$\tau^\sigma(g^\sigma d^\sigma) = \rho^\sigma(g^\sigma)\tau^\sigma(d^\sigma)$$

for all $g^\sigma \in G_a^\sigma$ and $d^\sigma \in D^\sigma$. Let G_0^σ and G'^σ_0 be the connected components of the identity of $Aut(D^\sigma)$ and $Aut(D'^\sigma)$ respectively. Using the fact that G_a^σ is dense in G_0^σ (see Proposition 1.3), we can construct the homomorphism ρ_1^σ in one of the main theorems contained in [10]

THEOREM 1.6. *There exist a finite covering G_1^σ of G_0^σ and a homomorphism $\rho_1^\sigma : G_1^\sigma \to G_0'^\sigma$ of Lie groups such that ρ_1^σ and τ^σ are equivariant and $\rho_1^\sigma(\Gamma^\sigma)$ is contained in Γ'^σ.*

Proof. See [10, Theorem 5.2].

2. Conjugates of symmetries

Let the symmetric domain $D = G/K$ and the arithmetic variety $X = \Gamma \setminus D$ be as in §1. For each $z \in D$, there is an involutive automorphism S_z of D, called a symmetry of D at z, such that z is an isolated fixed point of S_z and $S_z^2 = id$. If G_0 denotes the connected component of the identity of $Aut(D)$, then $S_z \in G_0$ for each $z \in D$ (see e.g. [16, Chapter II]). Given $\sigma \in Aut(\mathbf{C})$, let D^σ be the universal covering manifold of X^σ as before. In this section we shall consider the symmetries of D^σ associated to S_z for $z \in D$.

Let $\Gamma(X)$ (resp. $\Gamma(X^\sigma)$) be the fundamental group of X (resp. X^σ), and let $G_a(X)$ (resp. $G_a(X^\sigma)$) be the commensurability group of $\Gamma(X)$ (resp. $\Gamma(X^\sigma)$) in $Aut(D)$ (resp. $Aut(D^\sigma)$) as in §1. We set

$$\Delta(X) = \{z \in D \mid S_z \in G_a(X)\},$$

and similary set

$$\Delta(X^\sigma) = \{z \in D^\sigma \mid S_z \in G_a(X^\sigma)\}.$$

PROPOSITION 2.1. *Let $\sigma \in Aut(\mathbf{C})$. Then $\Delta(X)$ is nonempty if and only if $\Delta(X^\sigma)$ is nonempty.*

Proof. Since $(X^\sigma)^{\sigma^{-1}} = X$ and $(D^\sigma)^{\sigma^{-1}} = D$, it suffices to show that $\Delta(X^\sigma)$ is nonempty if $\Delta(X)$ is nonempty. Suppose $S_z \in G_a(X)$ for some $z \in D$. Consider a projective system $\{X_k \mid k = 1, 2, 3, \dots\}$ of finite unramified covering manifolds of X, where each $X_k = \Gamma_k \setminus D$ is an arithmetic variety. We set

$$X_{k,z} = (S_z \Gamma_k S_z \cap \Gamma_k) \setminus D$$

for each k. Then the collection $\{X_{k,z}\}$ is also a projective system of finite unramified covering manifolds of X. Since

$$S_z(S_z\Gamma_k S_z \cap \Gamma_k)d = (\Gamma_k S_z \cap S_z \Gamma_k)d = (\Gamma_k \cap S_z \Gamma_k S_z) S_z d$$

for each $d \in D$ and k, S_z induces the morphisms $S_{k,z} : X_{k,z} \to X_{k,z}$ such that $S_{k,z}$ is a symmetry of $X_{k,z}$ and

$$\pi_{k,z} \cdot S_{k,z} = S_{k,z} \cdot \pi_{k,z}$$

for each k; here the morphisms $\pi_{k,z} : X_{k,z} \to X$ are the natural covering maps. Applying σ to the morphisms $S_{k,z}$ and $\pi_{k,z}$, we obtain the morphisms

$$S^\sigma_{k,z} : X^\sigma_{k,z} \longrightarrow X^\sigma_{k,z}, \quad \pi^\sigma_{k,z} : X^\sigma_{k,z} \longrightarrow X^\sigma$$

such that each $S^\sigma_{k,z}$ is a symmetry of $X^\sigma_{k,z}$ and

$$\pi^\sigma_{k,z} \cdot S^\sigma_{k,z} = S^\sigma_{k,z} \cdot \pi^\sigma_{k,z}$$

for each k. Let $S_z^{(\sigma)} : D^\sigma \longrightarrow D^\sigma$ be the common lifting of the morphisms $S^\sigma_{k,z} : X^\sigma_{k,z} \to X^\sigma_{k,z}$. Then $S_z^{(\sigma)}$ is a symmetry on D^σ and $S_z^{(\sigma)} \in G_a(X^\sigma)$. Therefore there is a point $w \in D^\sigma$ with $S_z^{(\sigma)} = S_w$ and $S_w \in G_a(X^\sigma)$.

LEMMA 2.2. *Let $\mathcal{G}$ be a semisimple algebraic group defined over* $\mathbf{Q}$ *with trivial center such that the complex semisimple Lie group* $\mathcal{G}(\mathbf{C})$ *does not contain a factor isomorphic to either an exceptional group or the group* D_4 *in the sense of E. Cartan, and let* $\mathcal{D}$ *the symmetric space associated to the semisimple Lie group* $\mathcal{G}(\mathbf{R})$. *Then there is a symmetry of* $\mathcal{D}$ *contained in* $\mathcal{G}(\mathbf{Q})$.

Proof. This can be proved by using the embedding of $\mathcal{G}$ into the automorphism group of an algebra with involution constructed by A. Weil ([17]). See [11, Theorem 6] for details.

PROPOSITION 2.3. *Suppose that the algebraic group* $\mathbf{G}$ *considered in §1 satisfies the condition that the complex semisimple Lie group* $\mathbf{G}(\mathbf{C})$ *does not contain a factor isomorphic to either an exeptional group or the group* D_4. *Then* $\Delta(X)$ *is nonempty.*

Proof. The natural homomorphism $G \to Aut(D)$ induces the homomorphism $\alpha : \mathbf{G}(\mathbf{Q}) \to Aut(D)$. Since $\mathbf{G}$ has no factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$, ker α is the center of $\mathbf{G}(\mathbf{Q})$ and therefore finite. By Lemma 2.2 there is a symmetry on D contained in $\alpha(\mathbf{G}(\mathbf{Q}))$. Since $\alpha(\mathbf{G}(\mathbf{Q}))$ is contained in $G_a(X)$, it follows that $\Delta(X)$ is nonempty.

PROPOSITION 2.4. $\Delta(X^\sigma)$ *is dense in* D^σ.

Proof. It follows from Proposition 2.1 and Proposition 2.3 that $\Delta(X)$ is nonempty. Let $w \in \Delta(X^\sigma) \subset D^\sigma$. Then $S_w \in G_a(X^\sigma)$. If $h \in G_a(X^\sigma)$, then we have

$$S_{hz} = hS_z h^{-1} \in G_a(X^\sigma);$$

hence $hz \in \Delta(X^\sigma)$. Since $G_a(X^\sigma)$ is dense in G_0, the connected component of the identity, and G_0 acts on D^σ transitively, the set

$$\Delta_1(X^\sigma) = \{hz \mid h \in G_a(X^\sigma)\}$$

is dense in D^σ. Hence $\Delta(X^\sigma)$ is dense in D^σ.

3. The (H_1)-equivariance

In this section we show that for each $\sigma \in Aut(\mathbf{C})$ the pair $(\rho_1^\sigma, \tau^\sigma)$ constructed in [10] satisfies the (H_1)-equivariance which is weaker than

(H_2)-equivariance. Let (ρ, τ) be the equivariant pair considered in §1. Thus $\rho : G \to G'$ is a homomorphism of Lie groups and $\tau : D \to D'$ is a holomorphic map such that

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all} \quad g \in G \quad \text{and} \quad y \in D.$$

DEFINITION 3.1. (i) The equivariant pair (ρ, τ) is said to be (H_1)-equivariant if

$$\tau(S_z y) = S_{\tau(z)}\tau(y)$$

for all $y, z \in D$.

(ii) The equivariant pair (ρ, τ) is (H_2)- equivariant if

$$\rho(S_z) = S_{\tau(z)}$$

for all $z \in D$.

Obviously the (H_2)-equivariance implies the (H_1)-equivariance. We shall assume that (ρ, τ) is (H_2)-equivariant and that $\mathbf{G}(\mathbf{C})$ does not contain a factor isomorphic to either an exeptional group or the group D_4 as in Proposition 2.3. By Theorem 1.6 there exists a homomorphism $\rho_1^\sigma : G_1^\sigma \to G'^\sigma_0$ and $\tau^\sigma : D^\sigma \to D'^\sigma$ such that $(\rho_1^\sigma, \tau^\sigma)$ is an equivariant pair and $\rho_1^\sigma \subset \Gamma'^\sigma$. Let $\rho^\sigma : G_a^\sigma \to G_a(X'^\sigma)$ be the homomorphism described in §1 that induces the homomorphism ρ_1^σ (see [10, §4] for details).

PROPOSITION 3.2. *$\rho^\sigma(S_y) = S_{\tau^\sigma(y)}$ for all $y \in \Delta(X^\sigma)$.*

Proof. As in the proof of Proposition 2.4, $G_a(X^\sigma)$ contains a symmetry S_y of D^σ for some $y \in \Delta(X^\sigma)$; hence $G_a(X)$ contains a symmetry $S_z = S_y^{(\sigma^{-1})} \in G_a(X)$ of D for some $z \in D$ (see the proof of Proposition 2.1). Since

$$\rho(S_z) = S_{\tau(z)} \quad \text{and} \quad \rho(G_a(X)) \subset G_a(X'),$$

$\rho(S_z)$ is the symmetry $S_{\tau(z)}$ of D' with $\tau(z) \in \Delta(X')$. Using the construction contained in the proof of Proposition 2.1 once again, we obtain the symmetry $S_w = S_{\tau(z)}^{(\sigma)}$ of D' with $w \in \Delta(X'^\sigma)$. From the way ρ^σ is constructed in [10, §4] it follows that $S_w = \rho^\sigma(S_y)$. Thus we have

$$S_w\tau^\sigma(y) = \rho^\sigma(S_y)\tau^\sigma(y) = \tau^\sigma(S_y y) = \tau^\sigma(y)$$

for all $y \in \Delta(X^\sigma)$. Therefore we have

$$\rho(S_y) = S_{\tau^\sigma(y)} \quad \text{for all} \quad y \in \Delta(X^\sigma).$$

PROPOSITION 3.3. *The pair* $(\rho_1^\sigma, \tau^\sigma)$ *is* (H_1)*-equivariant.*

Proof. Given a projective system $\{X_k\}$ (resp. $\{X'_k\}$) of finite unramified covering manifolds of X (resp. X') and a point $z \in D$ (resp. $z' \in D'$), we set

$$X_{k,z} = (S_z \Gamma_k S_z \cap \Gamma_k) \setminus D,$$
$$X'_{k,z'} = (S_{z'} \Gamma'_k S_{z'} \cap \Gamma'_k) \setminus D'.$$

If S_y is a symmetry on D^σ at $y \in \Delta(X^\sigma)$, then as in the proof of Proposition 2.1 there is a symmetry $S_z = S_y^{(\sigma)}$ on D at $z \in \Delta(X)$. For each k the relation

$$\tau \cdot S_z = S_{\tau(z)} \cdot \tau$$

induces the following commutative diagram:

$$\begin{array}{ccc} X_{k,z} & \xrightarrow{\phi_{k,z}} & X'_{k,\tau(z)} \\ {\scriptstyle (S_z)_k}\downarrow & & \downarrow{\scriptstyle (S_{\tau(z)})_k} \\ X_{k,z} & \xrightarrow{\phi_{k,z}} & X'_{k,\tau(z)} \end{array}$$

Applying σ to this diagram, we obtain

$$\begin{array}{ccc} X^\sigma_{k,z} & \xrightarrow{\phi^\sigma_{k,z}} & X'^\sigma_{k,\tau(z)} \\ {\scriptstyle (S_z)^\sigma_k}\downarrow & & \downarrow{\scriptstyle (S_{\tau(z)})^\sigma_k} \\ X^\sigma_{k,z} & \xrightarrow{\phi^\sigma_{k,z}} & X'^\sigma_{k,\tau(z)} \end{array}$$

with $(S_z)^\sigma_k = (S_y)_k$ and $(S_{\tau(z)})^\sigma_k = (S_{\tau(y)})_k$. This induces the commutative diagram

$$\begin{array}{ccc} D^\sigma & \xrightarrow{\tau^\sigma} & D'^\sigma \\ {\scriptstyle S_y}\downarrow & & \downarrow{\scriptstyle S_{\tau^\sigma(y)}} \\ D^\sigma & \xrightarrow{\tau^\sigma} & D'^\sigma. \end{array}$$

Thus we have

$$\tau^\sigma \cdot S_y = S_{\tau^\sigma(y)} \cdot \tau^\sigma \quad \text{for all} \quad y \in \Delta(X^\sigma).$$

Since $\Delta(X^\sigma)$ is dense in D^σ, it follows that

$$\tau^\sigma \cdot S_y = S_{\tau^\sigma(y)} \cdot \tau^\sigma \quad \text{for all} \quad y \in D^\sigma.$$

4. The (H_2)-equivariance

Let (ρ, τ) be an (H_2)-equivariant pair as in §3, and let $(\rho_1^\sigma, \tau^\sigma)$ be as in Theorem 1.6. In this section we prove that $(\rho_1^\sigma, \tau^\sigma)$ is also (H_2)-equivariant. Let G_0 and G_0' be the connected components of the identity of $Aut(D)$ and $Aut(D')$ respectively. Then it is known that G_0 is generated by an even number of symmetries of D and that there exist a finite covering $\tilde{G}_0$ of G_0 and a homomorphism $\mu : \tilde{G}_0 \to G_0'$ such that

$$\mu(\pi^{-1}(S_{z_1} \cdots S_{z_k})) = S_{\tau(z_1)} \cdots S_{\tau(z_k)},$$

where $\pi : \tilde{G}_0 \to G_0$ is the covering map, k is a nonnegative even integer and $z_i \in D$ for each i (see [14, p. 426]). Thus, if G_0^σ and G'^σ_0 are as in §1, there exist a finite covering $\tilde{G}_0^\sigma$ of G_0^σ and a homomorphism $\tilde{\rho} : \tilde{G}_0^\sigma \to G'^\sigma_0$ such that

$$\tilde{\rho}(S_{y_1} \cdots S_{y_k}) = S_{\tau^\sigma(y_1)} \cdots S_{\tau^\sigma(y_k)};$$

here we identified $S_{y_1} \cdots S_{y_k}$ with its inverse image under the covering map $\tilde{G}_0^\sigma \to G_0^\sigma$. Since $\tau^\sigma \cdot S_y = S_{\tau^\sigma(y)} \cdot \tau^\sigma$ for all $y \in D^\sigma$ by Proposition 3.3, we obtain

$$\tau^\sigma(S_{y_1} \cdots S_{y_k} y) = S_{\tau^\sigma(y_1)} \cdots S_{\tau^\sigma(y_k)} \tau^\sigma(y)$$

$$\tilde{\rho}(S_{y_1} \cdots S_{y_k})$$

for all $y \in D^\sigma$. Hence $(\rho_1^\sigma, \tau^\sigma)$ is (H_1)-equivariant.

LEMMA 4.1. *Let $f ; \mathcal{D} \to \mathcal{D}'$ be a holomorphic map of symmetric domains such that $f \cdot S_x = S_{f(x)} \cdot f$, and let $\mathcal{G}$, $\mathcal{G}'$ be the connected components of the identity of $Aut(\mathcal{D})$, $Aut(\mathcal{D}')$ respectively.*

(i) There exist a finite covering $\mathcal{G}_1$ of $\mathcal{G}$ and a homomorphism $\alpha : \mathcal{G}_1 \to \mathcal{G}'$ such that (f, α) is equivariant, i.e.,

$$f(gx) = \alpha(g) f(x)$$

for all $g \in \mathcal{G}_1$ *and* $x \in \mathcal{D}$.

(ii) $f : \mathcal{D} \to \mathcal{D}'$ *is equivariant for at most one* $\alpha : \mathcal{G}_1 \to \mathcal{G}'$.

Proof. The proof is contained in [5, §2.2], which we shall sketch below. Let $\mathcal{G}_2 \subset \mathcal{G} \times \mathcal{G}'$ be the connected compont of the set of pairs (g, g') such that $f(gx) = g'f(x)$. Then the projection map $pr_1 : \mathcal{G} \to \mathcal{G}$ is surjective. Let $\mathcal{G}_1$ be the product of those simple factors of $\mathcal{G}_2$ which map nontrivially to $\mathcal{G}_1$. Then $\mathcal{G}_1$ is a finite covering of $\mathcal{G}$ and the projection map $pr_2 : \mathcal{G}_2 \to \mathcal{G}'$ induces the homomorphism $\alpha : \mathcal{G}_1 \to \mathcal{G}'$ such that (f, α) is equivariant. To prove the uniqueness we first note that $\mathcal{G}$ is isogenous to $\mathcal{G} \times \tilde{K}$, where

$$\tilde{K} = \{g' \in \mathcal{G}' \mid g'f = f\}.$$

Then K' is compact and (f, α) is equivariant if and only if $\{(g, \alpha(g)0\} \subset \mathcal{G}_2$. Since there are no homomorphisms from $\mathcal{G}_1$ to $\tilde{K}$, $\{(g, \alpha(g))\} = \mathcal{G}_1'$ and therefore α is unique.

THEOREM 4.2. *Assume that* G *satisfies the condition in Proposition 2.3, and let* $(\rho_1^\sigma, \tau^\sigma)$ *be as in Theorem 1.6. If* (ρ, τ) *is* (H_2)*-equivariant, then* $(\rho_1^\sigma, \tau^\sigma)$ *is also* (H_2)*-equivariant.*

Proof. If $\tilde{\rho} : \tilde{G}_0^\sigma \to G'^\sigma_0$ is as before, then by Lemma 4.1 we may assume that $\rho_1^\sigma = \tilde{\rho}$. Thus we have

$$\rho_1^\sigma(S_{y_1} \cdots S_{y_k}) = S_{\tau^\sigma(y_1)} \cdots S_{\tau^\sigma(y_k)}$$

for k even and $y_i \in D^\sigma$ for each i. Let G_2 be the subgroup of G_0^σ generated by all products of even number of symmetries of D^σ contained in $G_a(X^\sigma)$. Then G_3 is contained in $G_a(X^\sigma)$ and is dense in G_0^σ. By Proposition 3.2 $\rho^\sigma(S_y) = S_{\tau^\sigma(y)}$ for all $S_y \in G_a(X^\sigma)$. Thus we have

$$\rho_1^\sigma \mid_{G_3} = \rho^\sigma \mid_{G_3},$$

and hence

$$\rho_1^\sigma(S_y) = \rho^\sigma(S_y) = S_{\tau^\sigma(y)}$$

for all $S_y \in G_a(X^\sigma)$. Since $G_a(X^\sigma)$ is dense in G_0^σ, it follows that

$$\rho_1^\sigma(S_y) = S_{\tau^\sigma(y)} \quad \text{for all} \quad y \in D^\sigma.$$

COROLLARY 4.3. *Assume that* G *satisfies the condition in Proposition 2.3 and that* Γ *is cocompact. Let* $\pi : Y \to X$ *be a Kuga fiber variety associated to an* (H_2)*-equivariant pair* (ρ, τ)*. Then the conjugate* $\pi^\sigma : Y^\sigma \to X^\sigma$ *is also a Kuga fiber variety associated to the* (H_2)*-equivariant pair* $(\rho_1^\sigma, \tau^\sigma)$.

Proof. This is an immediate consequence of the Theorem 4.2 and [10, Theorem 6.3].

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Department of Mathematics
University of Northern Iowa
Cedar Falls, Iowa 50614, USA