

FIXED POINT THEOREMS IN METRIC SPACES WITH UNIFORMLY NORMAL STRUCTURE

JONG SOOK BAE AND MYOUNG SOOK PARK

1. Introduction

In 1965, Browder [4] proved that every nonexpansive mapping from a nonempty bounded, closed and convex subset K of a uniformly convex Banach space into K has a fixed point. Later, Goebel and Kirk [7] showed that for every uniformly convex Banach space there exists a constant $\gamma_0 > 1$ such that any uniformly γ -Lipschitzian mapping on K with $\gamma < \gamma_0$ has a fixed point. Furthermore, Casini and Maluta [5] proved that every Banach space X with uniformly normal structure has the fixed point property for uniformly γ -Lipschitzian mappings with $\gamma < \tilde{N}(X)^{-1/2}$.

On the other hand, it has always been tempting to generalize certain fixed point theorems to metric spaces. Many results on the fixed point property in metric spaces were developed after Penot's formulation [12]. The compactness of the convexity structure which appears in this formulation expresses the weak compactness, or more precisely, the reflexivity in the case of Banach spaces. In [8], Khamsi proved that a complete metric space with uniformly normal structure has a kind of intersection property, which is equivalent to the reflexivity in Banach spaces. Eventually this result generalizes Bae [3] and Maluta [11] to metric spaces.

In this paper we study some properties of metric spaces with uniformly normal structure. Moreover, we obtain a fixed point theorem for a complete metric space having a convexity structure \mathcal{F} with $\tilde{N}(\mathcal{F}) < 1$. Finally we prove a fixed point theorem for a nonexpansive inward mapping in a hyperconvex Banach space and extend the above theorem to the multivalued case.

Received December 30, 1991.

Supported by GARC, 1991 .

The notion of hyperconvex spaces is due to Aronszajn and Panitchpakdi [2] who proved that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. Sine [14] and Soardi [16] showed independently that a nonexpansive (point valued) mapping of a bounded hyperconvex space has a fixed point. Also Sine [15] showed that a ball intersection valued nonexpansive multivalued mapping on a hyperconvex space admits a nonexpansive point valued selection. This implies the existence of fixed points for such multivalued mappings.

Here, we introduce the convexity structure and the property (R). Let \mathcal{F} be a nonempty family of subsets of a metric space M . We say that \mathcal{F} defines a *convexity structure* on M if and only if \mathcal{F} is stable by intersection and contains the closed balls. An *admissible subset* of M is any intersection of closed balls. Let us denote the family of admissible subsets of M by $\mathcal{A}(M)$. And we say that \mathcal{F} has the *property (R)* if for any decreasing sequence $\{C_n\}$ of nonempty closed bounded subsets of M with $C_n \in \mathcal{F}$ has a nonempty intersection.

2. Properties of metric spaces with uniformly normal structure

Let M be a metric space. For a nonempty bounded subset C of M , we define

$$\begin{aligned} r(x, C) &= \sup\{d(x, y); y \in C\} \quad \text{for } x \in M, \\ \delta(C) &= \sup\{r(x, C); x \in C\} \quad \text{and} \\ R(C) &= \inf\{r(x, C); x \in C\} \end{aligned}$$

And we define the normal and uniformly normal structure in metric spaces. We say that a metric space M has normal structure [resp. *uniformly normal structure*] if there exists a convexity structure \mathcal{F} on M such that

$$R(C) < \delta(C) \quad [\text{resp. } \nabla R(C) \leq c\delta(C) \quad \text{for a fixed constant } c \in (0, 1)]$$

for any nonempty $C \in \mathcal{F}$, which is bounded and not reduced to a single point. We will also say that \mathcal{F} is normal [resp. uniformly normal]. Now we are able to define the *constant of the uniformity of normal structure* $\tilde{N}(\mathcal{F})$ for a convexity structure \mathcal{F} of a metric space M as

$$\tilde{N}(\mathcal{F}) = \sup\{R(C)/\delta(C); C \in \mathcal{F}, 0 < \delta(C) < \infty\}.$$

We note that a convexity structure on M is uniformly normal if $\tilde{N}(\mathcal{F}) < 1$.

In this paper we will always denote $\text{co}A$ by $\bigcap\{C; A \subseteq C, C \in \mathcal{F}\}$ and $\text{cl}A$ the closure, respectively, of a subset A of M .

LEMMA 2.1. *Let M be a metric space with a convexity structure \mathcal{F} and A be a nonempty bounded subset of M . Then $\delta(\text{co}A) = \delta(A)$.*

Proof. Obviously $\delta(A) \leq \delta(\text{co}A)$. Let x, y be in $\text{co}A$. Since $A \subset B(z, \delta(A))$ for any $z \in A$ and $B(z, \delta(A)) \in \mathcal{F}$, both x and y belong to $B(z, \delta(A))$, where $B(z, r) = \{u \in M; d(z, u) \leq r\}$, for $r \geq 0$, is the closed ball centered at z of radius r . This implies $A \subseteq B(x, \delta(A))$, so that $y \in \text{co}A \subseteq B(x, \delta(A))$. This implies $d(x, y) \leq \delta(A)$ and completes the proof.

THEOREM 2.2. *Let M be a complete metric space with a convexity structure \mathcal{F} which is uniformly normal. Let $\{C_n\}$ be a decreasing sequence of nonempty bounded subsets of M with $C_n \in \mathcal{F}$. Then $\bigcap_n \text{cl}C_n \neq \emptyset$.*

Proof. If $\delta(C_n) = 0$ for some $n \in N$, then it is nothing to prove. Hence, we assume $\delta(C_n) > 0$ for all $n \in N$. Let η be a real number with $\tilde{N}(\mathcal{F}) < \eta < 1$ and define a sequence $\{x_{kn}\}$ by induction: Take arbitrary $x_{1n} \in C_n$ for each $n \in N$. For $k \geq 2$, choose $x_{kn} \in \text{co}\{x_{k-1m}\}_{m \geq n}$ such that

$$\sup\{d(x_{kn}, x); x \in \text{co}\{x_{k-1m}\}_{m \geq n}\} \leq \eta\delta(\text{co}\{x_{k-1m}\}_{m \geq n}).$$

Since for every $k, n \in N$, $\text{co}\{x_{km}\}_{m \geq n} \subseteq C_n$ and for $m \geq n$,

$$\begin{aligned} d(x_{kn}, x_{km}) &\leq \sup\{d(x_{kn}, x); x \in \text{co}\{x_{k-1i}\}_{i \geq n}\} \\ &\leq \eta\delta(\{x_{k-1i}\}_{i \geq 1}), \end{aligned}$$

we have, for $k \geq 2$,

$$\delta(\{x_{kn}\}) \leq \eta\delta(\{x_{k-1n}\}).$$

Therefore we obtain, for $k \geq 2$,

$$\delta(\{x_{kn}\}) \leq \eta^{k-1}\delta(C_1).$$

Now we consider a subsequence $\{x_{nn}\}$ of $\{x_{kn}\}$. Then $\{x_{nn}\}$ is Cauchy, since $d(x_{nn}, x_{mm}) \leq \eta^{n-1}\delta(C_1)$ when $m \geq n$. Hence there exists an $x \in \bigcap_n \text{cl}C_n$ such that $\{x_{nn}\}$ converges to x . Therefore, $\bigcap_n \text{cl}C_n \neq \phi$.

The above theorem is a slightly different version of Bae [3] and Maluta [11] for metric spaces. In fact, they proved the reflexivity of a Banach space with uniformly normal structure independently. As an immediate consequence of Theorem 2.2 we have the following corollary which is due to Khamsi [8] by using the fixed point property for a nonexpansive mapping defined on a complete bounded metric sapce with a uniformly normal convexity structure.

COROLLARY 2.3 (KHAMSI [8]). *Let M be a complete metric space with a convexity structure \mathcal{F} which is uniformly normal. Then \mathcal{F} has the property (R).*

The proof is clear by Theorem 2.2. On the other hand, if we assume that $\text{cl}C \in \mathcal{F}$ whenever $C \in \mathcal{F}$, Theorem 2.2 easily follows from Corollary 2.3.

COROLLARY 2.4. *Let M be a complete metric space with a convexity structure \mathcal{F} which is uniformly normal. The for any nonempty closed bounded element C of \mathcal{F} , the set $A = \{x \in C; r(x, C) = R(C)\}$ is a nonempty closed element of \mathcal{F} .*

Proof. For each $n \in N$, we define

$$A_n = \{x \in C; r(x, C) \leq R(C) + (1/n)\}.$$

Then each A_n is nonempty and closed. Let $z \in \text{co}A_n$ and $y \in C$. Since $A_n \subseteq B(y, R(C) + (1/n))$, $\text{co}A_n \subseteq B(y, R(C) + (1/n)) \in \mathcal{F}$. Thus $d(x, y) \leq R(C) + (1/n)$, and hence $z \in A_n$. Therefore, we have $A_n =$

$\text{co}A_n \in \mathcal{F}$. By applying Theorem 2.2, we conclude that $A = \bigcap_n A_n$ is a nonempty closed element of \mathcal{F} .

3. A fixed point theorem for uniformly γ -Lipschitzian mappings

We say that a mapping T from a metric space M into itself is *uniformly γ -Lipschitzian* with $\gamma > 0$ if

$$d(T^n x, T^n y) \leq \gamma d(x, y)$$

for all $x, y \in M$ and each $n \geq 1$. A uniformly γ -Lipschitzian mapping is said to be *nonexpansive* if $\gamma = 1$. We also say that T has a *bounded orbit* at $x \in M$ if the orbit $\{T^n x\}$ of x is a bounded subset of M .

We say that a convexity structure on a metric space M is of *finite type* if for any $C \subseteq M$, $x \in \text{co}C$, and any $\epsilon > 0$, there are finite elements $x_1, \dots, x_n \in C$ and $y \in \text{co}\{x_1, \dots, x_n\}$ such that $d(x, y) < \epsilon$. For an example, a convexity structure consisting of all (closed) convex subsets of a normed space is of finite type.

In this section, we prove a fixed point theorem for a uniformly γ -Lipschitzian mapping T under the condition that T has a bounded orbit and $\gamma < \tilde{N}(\mathcal{F})^{-1/2}$ in a complete metric space having a convexity structure \mathcal{F} which is uniformly normal and of finite type. Our result extends a result of Casini and Maluta [5].

We define the asymptotic diameter and radius of a bounded sequence $\{x_n\}$ by

$$d_a(\{x_n\}) = \limsup_k \{d(x_m, x_n); m, n \geq k\}$$

and

$$r_a(x, \{x_n\}) = \limsup_n d(x, x_n).$$

The following lemma is a crucial tool to prove our theorem.

LEMMA 3.1. *Let M be a complete metric space with a convexity structure \mathcal{F} which is uniformly normal and of finite type. Then for every bounded sequence $\{x_n\}$, there exists a point z in $\text{clco}\{x_n\}$ such that*

(i) $r_a(z, \{x_n\}) \leq \tilde{N}(\mathcal{F})d_a(\{x_n\})$ and

(ii) for every $y \in M$, $d(z, y) \leq r_a(y, \{x_n\})$.

Proof. For each $n \geq 1$, set $A_n = \text{co}\{x_m\}_{m \geq n}$. By the definition, we can choose $z_n \in A_n$ such that

$$r(z_n, A_n) = \tilde{N}(\mathcal{F})\delta(A_n) + (1/n).$$

Then by Theorem 2.2. $\bigcap_n \text{clco}\{z_m\}_{m \geq n} \neq \emptyset$, and we claim that $z \in \bigcap_n \text{clco}\{z_m\}_{m \geq n}$ is the desired one. First note that $z \in \bigcap_n \text{cl}A_n \subseteq \text{clco}\{x_n\}$. To prove (i), let $d_a(\{x_n\}) = a$. Then for any $\epsilon > 0$, there exists a positive integer N such that $\delta(A_n) < a + \epsilon$ for all $n \geq N$. Take an arbitrary integer $n_0 \geq N$. Since we have $z \in \text{clco}\{z_m; n_0 \leq m\}$ and \mathcal{F} is of finite type, there exist a positive integer k and $y \in \text{co}\{z_m; n_0 \leq m \leq n_0 + k\}$ such that $d(y, z) < \epsilon$. For $n_0 \leq n \leq n_0 + k$, we have

$$\sup\{d(z_n, x_m); m \geq n_0 + k + 1\} \leq \tilde{N}(\mathcal{F})(a + \epsilon) + (1/n_0).$$

That is, for each $n_0 \leq n \leq n_0 + k$, $z_n \in B(x_m, \tilde{N}(\mathcal{F})(a + \epsilon) + (1/n_0))$ for all $m \geq n_0 + k + 1$. Therefore $y \in B(x_m, \tilde{N}(\mathcal{F})(a + \epsilon) + (1/n_0) + \epsilon)$ for all $m \geq n_0 + k + 1$. This implies that

$$r_a(z, \{x_n\}) \leq \tilde{N}(\mathcal{F})(a + \epsilon) + (1/n_0) + \epsilon.$$

Since ϵ and n_0 are arbitrary, we conclude that (i) holds. To show (ii) let $y \in M$ and $r_a(y, \{x_n\}) = \beta$. For any $\epsilon > 0$, there exists a positive integer N such that $d(y, x_n) \leq \beta + \epsilon$ for all $n \geq N$. Since $z \in \text{cl}A_N \subset B(y, \beta + \epsilon)$, we have $d(z, y) \leq \beta + \epsilon$. Since ϵ is arbitrary, (ii) holds.

THEOREM 3.2. *Let M be a complete metric space with a convexity structure \mathcal{F} which is uniformly normal and of finite type. Let $T : M \rightarrow M$ be a uniformly γ -Lipschitzian mapping with $\gamma < \tilde{N}(\mathcal{F})^{-1/2}$. Suppose that T has a bounded orbit at a point $y \in M$. Then T has a fixed point.*

Proof. Suppose that $\{T^n x\}$ is bounded for some $x \in M$ and $z(x)$ is the point satisfying Lemma 3.1 for $\{T^n x\}$. Set $r(x) = r(x, \{T^n x\})$. By (i) of Lemma 3.1 we have

$$\begin{aligned}
 r_a(z, \{T^n x\}) &\leq \tilde{N}(\mathcal{F})d_a(\{T^n x\}) \\
 &\leq \tilde{N}(\mathcal{F})\sup\{d(T^m x, T^n x); m, n \geq 0\} \\
 &\leq \tilde{N}(\mathcal{F})\gamma\sup\{d(T^n x, x); n \geq 0\} \\
 &\leq \tilde{N}(\mathcal{F})\gamma r(x).
 \end{aligned}$$

Moreover, we have for $N \geq 1$

$$\begin{aligned}
 r_a(T^N z, \{T^n x\}) &= \limsup_n d(T^N z, T^n x) \\
 &\leq \gamma \limsup_n d(z, T^{n-N} x) \\
 &= \gamma r_a(z, \{T^n x\}).
 \end{aligned}$$

Therefore, by combining the above inequalities, (ii) of Lemma 3.1 yields

$$r_a(z) \leq \gamma^2 \tilde{N}(\mathcal{F})r(x) = \eta r(x) \quad \text{with } \eta < 1.$$

Define a sequence $\{x_n\}$ by the following way; $x_1 = y$ and $x_{n+1} = z(x_n)$ for $n \geq 1$. Then $\{x_n\}$ is a Cauchy sequence. In fact, we have

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq d(x_{n+1}, T^k x_n) + d(T^k x_n, x_n) \\
 &\leq d(x_{n+1}, T^k x_n) + r(x_n).
 \end{aligned}$$

And by letting $k \rightarrow \infty$, we obtain

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq r_a(x_{n+1}, \{T^j x_n\}) + r(x_n) \\
 &\leq (1 + \gamma \tilde{N}(\mathcal{F}))r(x_n)
 \end{aligned}$$

By using induction we have

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq (1 + \gamma \tilde{N}(\mathcal{F}))r(x_n) \\
 &\leq (1 + \gamma \tilde{N}(\mathcal{F}))\eta^{n-1}r(x_1).
 \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy. Let $p = \lim x_n$. Then we have $Tp = p$ because

$$\begin{aligned} d(p, Tp) &\leq d(p, x_n) + d(x_n, Tx_n) + d(Tx_n, Tp) \\ &\leq (1 + \gamma)d(p, x_n) + r(x_n). \end{aligned}$$

4. Fixed point theorems in hyperconvex Banach spaces

A metric space M is said to be *hyperconvex* if it has the following intersection property. If $\{x_\alpha\}$ is a collection of points of M and $\{r_\alpha\}$ a corresponding collection of nonnegative real numbers with

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,$$

then the collection of closed balls has the nonempty intersection; that is,

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \phi.$$

In this section we prove a fixed point theorem for a nonexpansive inward mapping in a hyperconvex Banach space. Moreover, we extend the above theorem to the multivalued case.

Throughout this section we assume that the convexity structure on M will be the class $\mathcal{A}(M)$ of all admissible subsets of M . It is known that the constant of the uniformity of normal structure $\tilde{N}(\mathcal{A}(M))$ of a hyperconvex space M is $1/2$. However, Theorem 3.2 can not be applied to a hyperconvex space, even if T is nonexpansive (cf. [9]). As the following example shows, the convexity structure $\mathcal{A}(M)$ of a hyperconvex space M need not be of finite type. Therefore, the finite type assumption in Theorem 3.2 can not be removed.

EXAMPLE. (cf. [10]) Let λ denote the Banach limit in $\mathcal{L}_\infty(N)$ and define $T : \mathcal{L}_\infty(N) \rightarrow \mathcal{L}_\infty(N)$ by, for each $a = (a_1, a_2, \dots)$,

$$T(a) = (1 + \lambda(a), a_1, a_2, \dots).$$

Then the orbit of the origin is bounded, since

$$T^n(0, 0, \dots) = (1, \dots, 1, 0, \dots)$$

with 1 in the first n coordinates. It is easily checked that T is non-expansive and fixed point free. It is also checked that the convexity structure $\mathcal{A}(\mathcal{L}_\infty(N))$ is not of finite type and Lemma 3.1 does not hold. In fact, let $x_n = (1, \dots, 1, 0, \dots)$ with 1 in the first n coordinates. Then there exists a unique $z_n = (1, \dots, 1, 1/2, \dots) \in \text{co}\{x_m\}_{m \geq n}$ such that

$$r(z_n, \{x_m\}_{m \geq n}) = (1/2)\delta(\{x_m\}_{m \geq n}).$$

In this case we can check that

$$\{z = (1, 1, \dots)\} = \bigcap_n \text{clco}\{z_m\}_{m \geq n} = \bigcap_n \text{clco}\{x_m\}_{m \geq n}.$$

However, we have

$$r_a(z, \{x_n\}) = 1 = d_a(\{x_n\}),$$

even if $\tilde{N}(\mathcal{A}(\mathcal{L}_\infty(N)))$ is $1/2$.

LEMMA 4.1. *Let M be a hyperconvex space and $\{C_\alpha\}$ be a decreasing net in $\mathcal{A}(M)$ such that for any α , $C_\alpha \neq \phi$. Then $\bigcap_\alpha C_\alpha \neq \phi$.*

The proof directly follows from the definition of a hyperconvex space. Now we have our main result in this section. For a convex subset C of a Banach space X and $x \in C$, the *inward set* $I_C(x)$ is defined by

$$I_C(x) = \{(1 - \lambda)x + \lambda y; y \in C, \lambda \geq 0\}.$$

THEOREM 4.2. *Let X be a hyperconvex Banach space and C a nonempty bounded subset of X with $C \in \mathcal{A}(X)$, and let $f : C \rightarrow X$ be a nonexpansive inward mapping; that is, $fx \in I_C(x)$ for each $x \in C$. Then f has a fixed point.*

Proof. By the inwardness condition, for each $x \in C$ there exist a number $c(x) \geq 0$ and a point $gx \in C$ such that

$$fx = x + c(x)(gx - x).$$

As C is convex, by changing c and g if necessary, we can assume that $c(x) \geq 1$.

Consider the following nonempty family ordered by inclusion:

$$\mathcal{P} = \{D \subseteq C; D \neq \phi, D \in \mathcal{A}(X) \text{ and } gx \in I_D(x) \text{ for each } x \in D\}.$$

Then Lemma 4.1 and Zorn's Lemma supply us with a minimal element M of \mathcal{P} . Now we claim that M is a singleton. Suppose that M contains more than one point. Let $p \in M$ be a nondiametral point of M , that is, $r = r(p, M) < \delta(M)$. Let

$$Q = \cap \{B(x, r); x \in M\}.$$

Then Q is nonempty and $Q \in \mathcal{A}(X)$. Also $N = Q \cap M$ is a nonempty proper subset of M . We will show that $N \in \mathcal{P}$.

Indeed let $z \in N$ and consider $S = B(fz, r) \cap M$. Fix a point $u \in M$. Then a mapping $h : M \rightarrow X$ defined by $hx = u + t(fx - u)$ for all $x \in M$ has a fixed point provided $0 \leq t < 1$ (see [6] or [13]). Since $M \subseteq B(z, r)$, it follows readily that

$$\inf\{\|x - fz\|; x \in M\} \leq r.$$

Since $M \in \mathcal{A}(X)$, Lemma 4.1 implies

$$S = B(fz, r) \cap M = \cap_{t > r} (B(fz, t) \cap M) \neq \phi.$$

Also we have $gx \in S$ for each $x \in S$ because $f(M) \subseteq B(fz, r)$.

Thus we know that $S \in \mathcal{P}$, and hence $S = M$ by the minimality of M . This means that $fz \in Q$, and consequently we get $gz \in N$ and $N \in \mathcal{P}$. But this contradicts the minimality of M . Therefore M consists of exactly one point, say p . Since $gp \in M$, finally we have $p = gp = fp$. This completes the proof.

The following lemma which is proved in [15] is needed to prove our second result.

LEMMA 4.3. *Let F be a nonexpansive mapping from an arbitrary metric space to the class of nonempty ball intersections in a hyperconvex space. Then F admits a nonexpansive point valued selection.*

THEOREM 4.4. *Let X be a hyperconvex Banach space and C a nonempty bounded subset of X with $C \in \mathcal{A}(X)$, and let $F : C \rightarrow X$ be a multivalued nonexpansive inward mapping such that $Tx \in \mathcal{A}(X)$. Then F has a fixed point.*

Proof. We apply Lemma 4.3 to obtain a point valued nonexpansive selection f of F satisfying the inwardness condition on C . Theorem 4.2 yields the existence of a fixed point of f , which is also a fixed point of F .

References

1. A.G. Aksoy and M.A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York and Verlin, 1990.
2. N. Aronszajn and P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439.
3. J.S. Bae, *Reflexivity of a Banach space with uniformly normal structure*, Proc. Amer. Math. Soc. **90** (1984), 269–270.
4. F.E. Brower, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Soc. U.S.A. **54** (1965), 1041–1044.
5. E. Casini and E. Maluta, *Fixed point of uniformly lipschitzian mappings in spaces with uniformly normal structure*, Nonlinear Analysis T.M.A. **9** (1985), 103–108.
6. D. Downing, *Fixed points of weakly inward multis*, Nonlinear Analysis T.M.A. **10** (1986), 1261–1262.
7. K.Goebel and W.A. Kirk, *A fixed point theorem for transformation whose iterates have uniform lipschitz constant*, Studia Math. **47** (1973), 135–140.
8. M.A. Khamsi, *On Metric spaces with uniformly normal structure*, Proc. Amer. Math. Soc. **106** (1989), 723–726.
9. M.A. Khamsi and S. Reich, *Nonexpansive mappings and semigroups in hyperconvex spaces*, Math. Japonica **35** (1990), 467–471.
10. M.A. Khamsi, M. Lin and R. Sine, *On the fixed points of commuting nonexpansive maps in hyperconvex spaces*, Preprint.
11. E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J. Math. **111** (1984), 357–369.
12. J.P. Penot, *Fixed points without convexity*, Bull. Soc. Math. France, Memoire **60** (1979), 129–152.
13. S. Reich, *Fixed points of nonexpansive functions*, J. London Math. Soc. **7** (1973), 5–10.
14. R. Sine, *On nonlinear contraction semigroups in sup norm spaces*, Nonlinear Analysis, T.M.A. **3** (1979), 885–890.
15. R. Sine, *Hyperconvexity and nonexpansive multifunctions*, Trans. Amer. Math. Soc. **315** (1989), 754–767.

16. P. Suardi, *Existence of fixed points of nonexpansive mappings in certain Banach lattices*, Proc. Amer. Math. Soc. **73** (1979), 25–29.

Myong-Ji University
Yongin-eup, 449-800, Korea

Chungnam National University
Daejeon, 305-764, Korea