J. Korean Math. Soc. 30(1993), No. 1, pp. 25–39

A RESTRICTION THEOREM ON THE FOLLAND–STEIN SPACES

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1. The Heisenberg group

DEFINITION. The heisenberg group H^n is the lie group of real dimension $2n+1$, whose underlying space is $R \times C^n$, and whose group law is given by

$$(t,z)(t',z') = (t + t' + 2Imz\bar{z'}, z + z').$$

Its Lie algebra is generated by the left invariant vector fields X_j, Y_j, T, $j = 1, ..., n$, given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

It is easy to verify the following commutation relations:

$$[X_j, Y_k] = 4\delta_{j,k}T,$$

and all other brackets are zero. On H^n there are a family of dilations $\gamma_r(t,z)$ that gives rise to a one parameter group of automorphisms on H^n given by $\gamma_r(t,z) = (r^2t, rz)$.

The homogeneous dimension of H^n is $Q = 2n + 2$.(Folland[3]) We define the norm on H^n by

$$|(t,z)| = (t^2 + |z|^4)^{1/4}.$$

Received March 29, 1991. Revised September 4, 1992.
This work was supported by KNSF, 1990 .

2.The sublaplacian

Let

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

The operator $\mathcal{L}$ is homogeneous of degree 2, and $\mathcal{L}^t = \mathcal{L}$.
In the Euclidean space a fundamental solution to Δ is given by

$$E = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)}|x|^{2-n}, \quad n \geq 3.$$

Analogously the fundamental solution to $\mathcal{L}$ is given by

$$\phi = C(|z|^4 + t^2)^{-n/2}, \quad C = \frac{2^{2-2n}\pi^{n+1}}{\Gamma(n/2)^2}.$$

(See Folland[3]). Thus $\mathcal{L}$ is locally solvable. The convolution of two functions f and g in H^n is defined by

$$f * g(u) = \int_{H^n} f(v)g(v^{-1}u)dv$$

3. The Bessel potential and the spaces S_α^p

The definition of the Bessel potential is from Folland[3]. The principal tool is the diffusion semigroup H_t generated by -$\mathcal{L}$.

There is a unique semigroup $\{H_t,\ 0 < t < \infty\}$ of linear opertors on $L^1 + L^\infty$ satisfying the following:

(1) $H_t f = f * h_t$, where $h_t(x) = h(x,t)$ is C^∞ away from 0, and on $H^n \times (0,\infty)$, $\int_{H^n} h_t(x)dx = 1$ for all t. Also for all t and x, $h(x,t) \geq 0$ and $h(rx, r^2 t) = r^{-Q}h(x,t)$.

(2) If $u \in C_0^\infty$, then

$$lim_{t\to 0}||t^{-1}(H_t u - u) + \mathcal{L}u||_\infty = 0$$

(3) H_t is self adjoint, i.e.,$H_t|_{L^p} = H_t|_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$

(4)$f \geq 0 \Longrightarrow H_t f \geq 0$, $H_t 1 = 1$. Now if we extend $h(x,t)$ to be 0 for $t \leq 0$, then h is the fundamental solution to $\mathcal{L} + \frac{\partial}{\partial t}$.

Now inspired from the classical case, we define the Bessel potential

$$J_\alpha = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} h(x,t)dt$$

If $f \in L^p$, $1 < p < \infty$, then $(I+\mathcal{L})^{-\alpha/2} f = f * J_\alpha$

So define S^p_α to be the immage of L^p under the operator $(I+\mathcal{L})^{-\alpha/2}$. We have the following properties of J_α. (See [3])

(1) J_α is defined for all $x \neq 0$ and even for $x = 0$ when $\alpha > Q$. J_α is C^∞ away from 0.

(2) As $x \to 0$,

$$\begin{aligned} |J_\alpha(x)| &= O(|x|^{\alpha-Q}) \ \ if \ \alpha < Q \\ &= O(log\frac{1}{|x|}) \ \ if \ \alpha = Q \end{aligned}$$

(3) As $x \to \infty$, $|J_\alpha(x)| = O(|x|^{-N})$ for all N. Hence, $J_\alpha \in L^1$ for all $\alpha > 0$.

The spaces $\Lambda^{p,q}_\alpha(H^n)$ is defined to be the space of those functions in $L^p(H^n)$ for which the following quantity is finite.

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}} [\int_{H^n} |f(uv) - f(v)|^p dv]^{q/p} du < \infty, \quad 0 < \alpha < 1.$$

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}} [\int_{H^n} |f(uv) + f(uv^{-1}) - 2f(v)|^p dv]^{q/p} du) < \infty, \quad 1 \geq \alpha.$$

We prove the theorem when the restriction is one of the hyperplanes of $R_t \times R^{2n-1}$ of H^n and the case $\alpha \geq 1$ since the case $\alpha \leq 1$ is proved by Mekias [1]. First we want to introduce a theorem of Stein which has given the motivation for the study.

4. Restriction and Extention theorem for the spaces $L^p_\alpha(R^n)$

4.1. THEOREM (STEIN[7]). *(A). The restriction map* $R: L^p_\alpha(R^n) \to \Lambda^{p,p}_\beta(R^m)$ *is a bounded linear map as long as* $\beta = \alpha - (n-m)/p > 0$ *and* $1 < p < \infty$.

(B). Conversely, there exists an extention map E

$$E: \Lambda^{p,p}_\beta(R^m) \to L^p_\alpha(R^n) \quad \textit{such that}$$

$$R(E(g)) = g \quad \forall g \in \Lambda^{p,p}_\beta(R^m), \ \ \beta > 0, \ \ 1 < p < \infty.$$

Now we introduce our main theorem.

THEOREM (A'). *Let* $1 \leq \alpha < 2 + \frac{1}{p}$. *The restriction map*

$$R: \ S^p_\alpha(H^n) \to \Lambda^{p,p}_{\alpha-\frac{1}{p}}(R_t \times H^{n-1}),$$

where $R_t \times R^{2n-1}$ *is one of the hyperplanes* $\{t, x_1, ..., x_i = 0, ..., x_{2n}\}$ *is a bounded map.*

Proof. The proof is divided into two parts. At first , we will show that

$$g = R(f) \in L^p(R^{2n}), \ \ for \ f \in S^p_\alpha(H^n)$$

But this follows immediately from the inclusion relation $S^p_\alpha \subset S^p_{\alpha'}$ for $\alpha > \alpha'$ (Folland[3]) once we show that $g = R(f)$ belongs to $L^p(R^n)$ for $f \in S^p_\alpha(H^n)$, $\alpha < 1$. For this we will adopt a proof from Stein[8] and Mekias[1].

After that we will try to show that $g \in \Lambda^{p,p}_{\alpha-\frac{1}{p}}(R_t \times H^{n-1})$.

Let's prove the first claim when $\alpha < 1$. Let $f \in S^p_\alpha(H^n)$. Then there is $\phi \in L^p(R^{2n+1})$ such that

$$f = \phi * J_\alpha, \ \ ||f||_{p,\alpha} = ||\phi||_p$$

We will restrict ourselves to the case n=1 just for simplicity. Put

$$g(t,y) = \int_{R^3} \phi(s,x',y') J_\alpha(t - s + 2yx', -x', y - y') ds dx' dy'$$

To prove that $g \in L^p(R^2)$ it suffices to show that

$$\int g(t,y)h(t,y)dtdy$$

is bounded for all $h \in L^q(R^2)$ where $\frac{1}{p}+\frac{1}{q}=1$ so that $||g||_{L^p}$ is finite.

$$|g(t,y)| \leq \int |\phi(s,x',y')||J_\alpha(t-s+2yx',-x',y-y')|dsdx'dy'$$

Let $h(t,y) \in L^q(R^2)$. Our claim is to show that

$$||g||_{L^p} = sup_{||h||_{L^q}\leq 1} \int_{R^2} |g(t,y)||h(t,y)|dtdy < \infty \quad here$$

$$\int_{R^2} |g(t,y)||h(t,y)|dtdy \leq \int_{R^2}\int_{R^3} |h(t,y)||\phi(s,x',y')|| J_\alpha(t-s+2yx',-x',y-y')|dsdx'dy'dtdy$$

Using Fubini's theorem the above equals

$$\int_{R^3} |\phi(s,x',y')\{| \int_{R^2} |h(t,y)||J_\alpha(t-s+2yx',-x',y-y')|dtdy\}dsdx'dy'$$

Let's study the inner integral

$$\int_{R^2} |h(t,y)||J_\alpha(t-s+2yx',-x',y-y')|dtdy$$

Fix x' and let

$$T_{x'}h(s,y') = \int_{R^2} h(t,y)J_\alpha(t-s+2yx',-x',y-y')dtdy$$

If we could show that

(1) $$sup_{s,y'} \int |J_\alpha(t-s+2yx',-x',y-y')|dtdy$$

and

$$(2) \qquad sup_{t,y} \int |J_\alpha(t-s+2yx', -x', y-y')| ds dy$$

are both bounded, independently of (t,y) and (s,y) respectively, then we could use Young's Inequality to conclude

$$||T_{x'}h||_q \leq C(x')||h||_q.$$

Consider (1) first

$$sup_{s,y'} \int |J_\alpha(t-s+2yx', -x', y-y')| dt dy$$

By the change of variables

$$t-s+2yx' = \tau, \quad y-y' = \eta$$

the above equals

$$(3) \qquad Int = \int_{R^2} |J_\alpha(\tau, -x', \eta)| d\tau d\eta$$

The estimate for this integral will be made for two separate cases (a) $|x'|$ small ($|x| \leq 1$), (b) $|x'|$ large ($|x'| > 1$). First we will estimate (a) and show $Int \leq C|x'|^{\alpha-1}$. For this we devide the above integral into two parts.

$$\int_{|\tau|^{1/2}+|x'|+|\eta|<1} |J_\alpha(\tau, -x', \eta)| d\tau d\eta + \int_{|\tau|^{1/2}+|x'|+|\eta|>1} |J_\alpha(\tau, -x', \eta)| d\tau d\eta$$
$$= I_1 + I_2.$$

Let's look at the main part I_1 first. (There is a fast decrease for I_2)

$$\begin{aligned} I_1(x') &= \int_{|\tau|^{1/2}+|x'|+|\eta|<1} |J_\alpha(\tau, -x', \eta)| d\tau d\eta \\ &\leq C \int_{|\tau|^{1/2}+|x'|+|\eta|<1} (|\tau|^{1/2} + |x'| + |\eta|)^{\alpha-Q} d\tau d\eta \\ &\leq C \int_{|\tau|^{1/2}+|\eta|<1, |x'|<1} (|\tau|^{1/2} + |x'| + |\eta|)^{\alpha-Q} d\tau d\eta \\ &\leq (i) + (ii) \qquad where \end{aligned}$$

$$
\begin{aligned}
(i) &= \int_{|\tau|^{1/2}+|\eta|<|x'|<1} (|\tau|^{1/2} + |x'| + |\eta|)^{\alpha-Q} d\tau d\eta \\
&\leq \int_{|\tau|^{1/2}+|\eta|<|x'|} |x'|^{\alpha-Q} d\tau d\eta \\
&\leq C|x'|^{\alpha-Q} \int_{|\tau|<|x'|^2, |\eta|<|x'|} d\tau d\eta \\
&= C|x'|^{\alpha-Q+3} = C|x'|^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
(ii) &= \int_{|\tau|^{1/2}+|\eta|>|x'|} (|\tau|^{1/2} + |x'| + |\eta|)^{\alpha-Q} d\tau d\eta \\
&\leq C\sum_{j=0}^{\infty} \int_{|\tau|^{1/2}+|\eta|<2^j|x'|} (|\tau|^{1/2} + |x'| + |\eta|)^{\alpha-Q} d\tau d\eta \\
&\leq C\sum_{j=0}^{\infty} \int_{|\tau|^{1/2}+|\eta|<2^j|x'|} (2^j|x'|)^{\alpha-Q} d\tau d\eta \\
&\leq C\sum_{j=0}^{\infty} (2^j|x'|)^{\alpha-Q} \int_{|\tau|<(2^j|x'|)^2, |\eta|<(2^j|x'|)} d\tau d\eta \\
&\leq C\sum_{j=0}^{\infty} (2^j|x'|)^{\alpha-Q} (2^j|x'|)^3 \\
&= C|x'|^{-1+\alpha} \sum_{j=0}^{\infty} (2^j)^{\alpha-1}
\end{aligned}
$$

Since $\alpha < 1$, we see that $\sum_{j=0}^{\infty}(2^j)^{\alpha-1} \leq \infty$.

So $I_1(x') \leq C|x'|^{-1+\alpha}$.

Now look at $I_2(x') = \int_{|\tau|^{1/2}+|x'|+|\eta|>1} |J_\alpha(\tau, -x', \eta)| d\tau d\eta$

Let N be any arbitrarily large number which is chosen as $N > 4 - \alpha$.

Then

$$\begin{aligned}
I_2(x') &\le C\int_{|\tau|^{1/2}+|x'|+|\eta|>1}(|\tau|^{1/2}+|x'|+|\eta|)^{-N}d\tau d\eta \\
&\le C\int_{|\tau|^{1/2}+|x'|+|\eta|>1,|\tau|^{1/2}+|\eta|<|x'|}(|\tau|^{1/2}+|x'|+|\eta|)^{\alpha-4}d\tau d\eta \\
&+C\int_{|\tau|^{1/2}+|x'|+|\eta|>1,|\tau|^{1/2}+|\eta|>|x'|}(|\tau|^{1/2}+|x'|+|\eta|)^{\alpha-4}d\tau d\eta \\
&\le C|x'|^{\alpha-4}+C\int_{|\tau|^{1/2}+|\eta|>|x'|}(|\tau|^{1/2}+|x'|+|\eta|)^{\alpha-4}d\tau d\eta
\end{aligned}$$

By subdividing the range $|\tau|^{1/2}+|\eta|>|x'|$ into shells

$$2^j|x'|<|\tau|^{1/2}+|x'|+|\eta|<2^{j+1}|x'|$$

and following similar argument as in the previous case(ii) the last integral can be shown to be bounded by $C'|x'|^{\alpha-4}$. So we have showd that

$$I_2(x')\le C''|x'|^{-1+\alpha} \quad and$$

$$\int_{R^2}|J_\alpha(t-s+2yx',-x',y-y')|dtdy\le C|x'|^{-1+\alpha} \quad |x'|\le 1$$

Now let's study (2) and estimate the integral, i.e., show

$$sup_{t,y}\int_{R^2}|J_\alpha(t-s+2yx',-x',y-y')|dsdy'\le C|x'|^{-1+\alpha}$$

But by the change of variables,

$$\int_{R^2}|J_\alpha(t-s+2yx',-x',y-y')|dsdy'=\int_{R^2}|J_\alpha(\tau,-x',z)|d\tau dz$$

So exactly same computation as before yields

$$\int_{R^2}|J_\alpha(\tau,-x',z)|d\tau dz\le C|x'|^{-1+\alpha}$$

Hence, applying Young's inequality we see that

$$||T_{x'}h||_q \leq C|x'|^{-1+\alpha}||h||_q$$

From this we can conclude

$$\begin{aligned}\int_{R^2} |g(t,y)||h(t,y)|dtdy &\leq \int_{R^3} |\phi(s,x',y')||T_{x'}h(s,y')|dsdx'dy' \\ &\leq ||\phi||_p(\int |T_{x'}h(s,y')|^q dsdx'dy')^{\frac{1}{q}} \\ &\leq ||\phi||_p(\int_{-\infty}^{\infty}[\int_{R^2} |T_{x'}h(s,y')|^q dsdy']dx')^{\frac{1}{q}} \\ &\leq ||\phi||_p||h||_q(\int |x'|^{(-1+\alpha)q}dx')^{\frac{1}{q}} < \infty\end{aligned}$$

if $\int_{-\infty}^{\infty} |x'|^{(-1+\alpha)q}dx'$ is integrable near 0 if $(-1+\alpha)q > -1$ i.e. $\alpha > \frac{1}{p}$.

Next we will estimate the case (b). For this we go back and adopt the estimate of I_2(since $|x'| > 1$): Choose N' be arbitrary large number such that

$$(-N'+3)q < -1$$

Following same steps there it can be shown that

$$Int \leq C|x'|^{-N'+3} < C|x'|^{-\frac{1}{q}}$$

In this case we can also show that

$$\int_{R^2} |g(t,y)||h(t,y)|dtdy \leq ||\phi||_p||h||_q(\int |x'|^{(-N'+3)q}dx')^{\frac{1}{q}} < \infty$$

We have finally showed that $g \in L^p(R^2)$.

From now on we will show that $g \in \Lambda^{p,p}_{\alpha-\frac{1}{p}}(R^2)$, i.e., we want to show that

$$\int\int \frac{|g(uv)+g(u^{-1}v)-2g(v)|^p}{|u|^{2+\alpha p}}dvdu < \infty$$

First note that when $|u| > 1$, the integral converges because

$$\int_{|u|>1}\int_{R^2}\frac{|g(uv)+g(u^{-1}v)-2g(v)|^p}{|u|^{2+\alpha p}}dvdu$$
$$\leq C\int_{|u|>1}\frac{1}{|u|^{2+\alpha p}}[\int|g(v)|^p dv]du$$
$$= ||g||_p^p\int_{|u|>1}\frac{1}{|u|^{2+\alpha p}}du$$

and

$$\int_{|u|>1}\frac{1}{|u|^{2+\alpha p}}du < \infty \quad since\ \alpha p > 1$$

So it remains to show that

$$\int_{|u|<1}\int_{R^2}\frac{|g(uv)+g(u^{-1}v)-2g(v)|^p}{|u|^{2+\alpha p}}dvdu < \infty$$

Set $u = (t,0,y), v = (\tau,0,z), w = (s,x',y')$. The function $g(t,y)$ can be written as

$$g(t,y) = \int_{H^1}\phi(s,x',y')J_\alpha(t-s+2yx',-x',y-y')dsdx'dy'.$$

which can be expressed as $g(t,y) = \int_{-\infty}^{\infty} g_{x'}(t,y)dx'$, where

$$g_{x'}(t,y) = \int_{R^2}\phi(s,x',y')J_\alpha(t-s+2yx',-x',y-y')dsdy'.$$

Now look at the quantity

$$|g_{x'}(t+\tau,y+z)+g_{x'}(-t+\tau,-y+z)-2g_{x'}(\tau,z)|$$
$$\leq \int|\phi(s,x',y')||J_\alpha(t+\tau-s+2x'(y+z),-x',y+z-y')$$
$$+J_\alpha(-t+\tau-s+2x'(-y+z),-x',-y+z-y')$$
$$-2J_\alpha(\tau-s+2zx',-x',z-y')|dsdy'$$

Let

$$\begin{aligned}K_{x',t,y}(s,y',\tau,z) = {} & J_\alpha(t+\tau-s+2x'(y+z),-x',y+z-y') \\ & + J_\alpha(-t+\tau-s+2x'(-y+z),-x',-y+z-y') \\ & - 2J_\alpha(\tau-s+2zx',-x',z-y')\end{aligned}$$

So if we could show that

$$sup_{\tau,z}\int_{R^2}|K_{x',t,y}(s,y',\tau,z)|dsdy' \tag{4}$$

$$sup_{s,y'}\int_{R^2}|K_{x',t,y}(s,y',\tau,z)|d\tau dz \tag{5}$$

are both dominated by a constant $C(x't,y)$, then we could conclude

$$\begin{aligned}&||g_{x'}(t+*,y+*)+g_{x'}(-t+*,-y+*)-2g_{x'}(*,*)||_p \\ &\qquad \le C(x',t,y)||\phi(*,x',*)||_p,\end{aligned}$$

and thus by the generalized Young's inequality

$$\begin{aligned}&||g(t+*,y+*)+g(-t+*,-y+*)-2g(*,*)||_p^p \\ &\qquad \le [\int C(x',t,y)||\phi(*,x',*)||_p dx']^p.\end{aligned}$$

On making the change of variables

$$\tau-s+2zx'\to\tau,\quad z-y'\to z$$

expression (5) becomes

$$\begin{aligned}\int_{R^2}|J_\alpha(t+\tau+2yx',-x',y+z)+J_\alpha(-t+\tau-2yx',-x',-y+z) \\ -2J_\alpha(\tau,-x',z)|d\tau dz\end{aligned}$$

On the other hand, by the change of variables

$$\tau-s+2zx'\to\tilde{s},\quad z-y'\to\tilde{y'}$$

expression (4) becomes

$$\int_{R^2} |J_\alpha(t+\tilde{s}+2x'y, -x', y+\tilde{y'}) + J_\alpha(-t+\tilde{s}-2x'y. -x', -y+\tilde{y'}) - 2J_\alpha(\tilde{s}, -x', \tilde{y'})| d\tilde{s} d\tilde{y'}$$

Since they are equivalent, let's estimate (5)

$$\int_{R^2} |K_{x',t,y}(s,y',\tau,z)| d\tau dz = \int_{R^2} |J_\alpha(t+\tau+2yx', -x', y+z) + J_\alpha(-t+\tau-2yx', -x', -y+z) - 2J_\alpha(\tau, -x', z)| d\tau dz.$$

Setting $u=(t,0,y)$, $v=(\tau,-x',z)$, the integrand becomes

$$\begin{aligned}
&|J_\alpha(uv) + J_\alpha(u^{-1}v) - 2J_\alpha(v)| \\
&\leq |u| sup_{\rho\in[0,1]} |\nabla J_\alpha((\rho u)v) - \nabla J_\alpha((\rho u^{-1})v)| \\
&= |u| sup_{\rho\in[0,1]} |\nabla J_\alpha(\rho^2 t + \tau + 2\rho y x', -x'z + \rho y) \\
&\quad - \nabla J_\alpha(\tau - \rho^2 t - 2\rho y x', -x', z - \rho y)| \\
&= |u| sup_{\rho\in[0,1]} \left| \int_0^1 \frac{d}{ds} \nabla J_\alpha(\tau + \rho^2 t + 2\rho y x' - 2s\rho^2 t \right. \\
&\quad \left. - 4s\rho y x', -x', z + \rho y - 2s\rho y) ds \right| \\
&= |u| sup_{\rho\in[0,1]} \left| \int_0^1 (2\rho^2 t + 4\rho y x', 0, 2\rho y) \nabla^2 J_\alpha(\tau + \rho^2 t + 2\rho y x' \right. \\
&\quad \left. - 2s\rho^2 t - 4s\rho y x', -x', z + \rho y - 2s\rho y) ds \right| \\
&\leq |u| sup_{\rho\in[0,1]} |(2\rho^2 t + 4\rho y x', 0, 2\rho y)| \left| \int_0^1 \nabla^2 J_\alpha(\tau + \rho^2 t + 2\rho y x' \right. \\
&\quad \left. - 2s\rho^2 t - 4s\rho y x', -x', z + \rho y - 2s\rho y) ds \right|
\end{aligned}$$

From now on, we will focus on $|x'| < 1$, since when $|x'| > 1$, we have rapid decrease of J_α and hence no problems at ∞.

Let $A = |(2\rho^2 t + 4\rho y x', 0, 2\rho y)| = |2\rho^2 t + 4\rho y x'|^{1/2} + |2\rho y|$.

When $|t|^{1/2} + |y|$ is small compared to $|x'|$, e.g., say $|t|^{1/2}+|y|<\frac{1}{100}|x'|$, then

$$A=|(2\rho^2t+4\rho yx',o,2\rho y)|<C(|t|^{1/2}+|y|)=C|u|,$$

and the quantity

$$|\tau+\rho^2t+2\rho yx'-2s\rho^2t-4s\rho yx'|^{1/2}+|x'|+|z+\rho y-2s\rho y|$$

is comparable to

$$|\tau|^{1/2}+|x'|+|z|,$$

and so

$$\begin{aligned}
&\int_{R^2}|K_{x',t,y}(s,y',\tau,z)|d\tau dz\\
&\quad\le C(|t|^{1/2}+|y|)^2\int_{R^2}(|\tau|^{1/2}+|x'|+|z|)^{-Q+\alpha-2}d\tau dz\\
&\le C(|t|^{1/2}+|y|)^2\int_{|\tau|^{1/2}+|z|<|x'|}(|\tau|^{1/2}+|x'|+|z|)^{-Q+\alpha-2}d\tau dz\\
&\quad+C(|t|^{1/2}+|y|)^2\int_{|\tau|^{1/2}+|z|>|x'|}(|\tau|^{1/2}+|x'|+|z|)^{-Q+\alpha-2}d\tau dz\\
&\le C(|t|^{1/2}+|y|)^2|x'|^{-3+\alpha}
\end{aligned}$$

For the last inequality to hold we need $-Q+\alpha-2<-3$. But with our choice of $\alpha<2+\frac{1}{p}<3$, this holds all the time.

When $|t|^{1/2}+|y|>\frac{1}{100}|x'|$, then we use triangle inequality and estimates on J_α as in the case of L^p-estimaate of g to get

$$\int_{R^2}|K_{x',t,y}(s,y',\tau,z)|d\tau dz\le C|x'|^{-1+\alpha}$$

Hence,

$$\begin{aligned}
&\int_{|u|<1}\frac{||g(t+*,y+*)+g(-t+*,-y+*)-2g(*,*)||_p^p}{|u|^{2+\alpha p}}du\\
&\quad\le C\int_{|u|<1}\frac{1}{|u|^{2+\alpha p}}(\int_{|x'|>100|u|}|u|^2|x'|^{-3+\alpha}||\phi(*,x',*)||_p dx')^p du\\
&\qquad+C\int_{|u|<1}\frac{1}{|u|^{2+\alpha p}}(\int_{|x'|<100|u|}|x'|^{-1+\alpha}||\phi(*,x',*)||_p dx')^p du
\end{aligned}$$

Now passing to polar coordinates by setting $|u| = r = |t|^{1/2} + |y|$, we see that the above inequality is bounded by

$$C\int_{r<1}(\int_{|x'|>100r}|x'|^{-3+\alpha}r^{2-\alpha}||\phi(*,x',*)||_p dx')^p dr$$

$$+C\int_{r<1}(\int_{|x'|<100r}|x'|^{-1+\alpha}r^{-\alpha}||\phi(*,x',*)||_p dx')^p dr$$

We see the function $K(x',r) = |x'|^{-1+\alpha}r^{-\alpha}$ is homogeneous of degree -1. Then after some computation and application of Young's inequality, we obtain

$$\int_{|u|<1}\frac{||g(t+*,y+*)+g(-t+*,-y+*)-2g(*,*)||_p^p}{|u|^{2+\alpha p}}du$$
$$\leq C(A^p + A'^p)||\phi||_p^p$$

where

$$A = \int_{100}^{\infty}|K(1,x')|x'^{-1/p}dx' = \int_{100}^{\infty}|x'|^{-3+\alpha}|x'|^{-1/p}dx'$$
$$= \int_{100}^{\infty}|x'|^{-3+(\alpha-\frac{1}{p})}dx'$$

which is finite since $\alpha - \frac{1}{p} < 2$.

$$A' = \int_{0}^{100}|K(1,x')|x'^{-1/p}dx' = \int_{0}^{100}|x'|^{-1+\alpha}|x'|^{-1/p}dx'$$
$$= \int_{0}^{100}|x'|^{-1+(\alpha-\frac{1}{p})}dx'$$

which is finite since $-1 + (\alpha - \frac{1}{p}) > -1$. Hence we showed that

$$g \in \Lambda^{p,p}_{\alpha-\frac{1}{p}}(R^2)$$

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