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Hessian Comparison Theorem of distance functions relative to some boundary points

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1. Introduction

Given two complete Riemannian manifolds and distance functions relative to some fixed points respectively, Green. Wu[1], Siu-Yau[2], S. T. Yau and Schoen[3] proved the Hessian comparision threorem of distance functions at point which is not conjugate point. In this paper, we show the Hessian comparision theorem of distance functions relative to some fixed points which are boundary points of each manifold.

Let M be an m-dimensional Riemannian manifold with boundary ($m \geq 2$). Let $\sigma(t),\ 0 \leq t \leq b$ be a geodesic in M parametrized by arc length with inital point $p \in \partial M$ and initial tangent $\sigma_*(0)$ orthogonal to the tangent space $(\partial M)_p$ The second fundamental form assigns the normal vector $\sigma_*(0)$ to a symmetric linear transformation $S_{\sigma_*(0)}$ on $(\partial M)_p$ Let $(\sigma,\ b,\ \partial M)$ denote the linear space of all piecewise smooth vector fields along $\sigma|[0,b]$ whose values are everywhere orthogonal to the tangent vector σ_* of σ and whose initial value is an element of $T(\partial M)$. A smooth vector field $Y(t)$ along σ is called a Jacobi field if it satisfies the Jacobi equation

$$(\nabla^2 Y)(t) + R_t Y(t) = 0$$

where $R_t Y(t)$ is $R(Y(t), \sigma_*(t))\sigma_*(t)$. A Jacobi field $Y \in (\sigma, b, \partial M)$ is called a(strong) ∂M -Jacobi field if it satisfies the boundary condition

$$S_{\sigma_*(0)} Y(0) = (\nabla Y)(0)$$

A focal point on σ is a point $\sigma(t), t \neq 0$ at which a nontrial ∂M -Jacobi field along σ vanishes. The index forn $I(b, \partial M)$ is a symmetric bilinear form on $\mathcal{L}(\sigma, b, \partial M)$ defined as follows :

$$+\int_0^t < \nabla X, \nabla Y > - < R_t X, Y > dt.$$

MINIMIZATION THEOREM1.1[4]. *Let* $X \in (\sigma, b, \partial M)$. *Assume that there are no focal points on* σ. *Then there exists a unique* ∂M *-Jacobi field Y such that*

$$Y(b) = X(b) \text{ and } I(X,X) \geq I(Y,Y)$$

with equality in case that $X = Y$.

Let N be an n-dimensional Riemannian manifold ($n \geq 2$). Let $\tau(t), 0 \leq t \leq b$ be a geodesic in N parametrized by arc length with initial point $q \in \partial N$
and initial tangent $\tau_*(0)$ orthogonal to the tangent space $(\partial N)_q$.

THEOREM1.2[4]. *Assume that the sectional curvature* $K(p)$ *and* $K(Q)$ *satisfy* $K(p) \leq K(Q)$ *for each* $t \in [0,b]$ *and for all 2-plane* $P \subset M_{\sigma(t)}$ *containing* $\sigma_*(t)$ *and all 2-planes* $Q \subset N$ *containing* $\tau_*(t)$ *and the second fundamental form elements of* ∂M *is the greater than second fundamental form elements of* ∂N. *Then, if there are no focal points on* τ, *there are no focal points on* σ

2. Comparison Theorem

THEOREM2.1. *Let M and N be compact Riemannian manifolds with boundary such that* $\dim M \leq \dim N$. *Let p and q be boundary points if M and N respectively. Let* ρ_M *and* ρ_N *be distance functions on M and N relative to P and q respectively. Let* $\gamma_1 : [0,b] \to M$ *and* $\gamma_2 : [0,b] \to N$ *be normal*
geodesics with $\gamma_1(0) = p$, $\gamma_2(0) = q$, $\gamma_{1*(0)} \in (\partial M)_p^\perp$ *and* $\gamma_{2*(0)} \in (\partial N)_q^\perp$ *If each radial curvature at* $\gamma_2(t)$ *is the greater than every radial curvature at* $\gamma_1(t)$ *and there are no focal points on* γ_2. *Then*

(1) $$D^2 \rho_N(Y,Y) \leq D^2 \rho_M(X,X)$$

for all $Y \in N_{\gamma 2(t)}$ *and for all* $X \in M_{\gamma 1(t)}$ *such that* $|X| = |Y|$ *and* $< X, \gamma_{1*(t)} > = < Y, \gamma_{2*(t)} >$

PROOF : It suffices to prove (1) For *t=b*. for $X \perp \gamma_{1*}(b)$,

$$\begin{aligned} D^2 \rho_M(\gamma_{1*}(b), X) &= D^2 \rho_M(X, \gamma_{1*}(b)) \\ &= X(\gamma_{1*}\rho M) - (D_X \gamma_{1*})(\rho M) \end{aligned}$$

$$= -< D_{X\gamma 1*}, grad\rho M >$$
$$= -< D_X\gamma_{1*}, \gamma_{1*} >= -\frac{1}{2}X < \gamma_{1*}, \gamma_{1*} >= 0$$

Similarly, for $Y \perp \gamma_{2*}(b)$, we have

$$D^2_{\rho N}(\gamma_{2*}(b), Y) = 0$$

On the other hand,

$$D^2_{\rho M}(\gamma_{1*}(b), \gamma_{1*}(b)) = \gamma_{1*}(b)(\gamma_{1*}\rho M) - (D_{\gamma 1*}\gamma_{1*})\rho M = 0$$

Similarly,

$$D^2_{\rho N}(\gamma_{2*}(b), \gamma_{2*}(b)) = 0$$

It remains to prove(1) when $X_1 \perp \gamma_{1*}(b)$ and $X_2 \perp \gamma_{2*}(b)$.

Let $\tilde{X}_i (i = 1,2)$ be the parallel vector field of X_i along γ_i By Theorem 1.1, there exist a unique ∂M-Jacobi field W_1 and a unique ∂N-Jacobi field W_2 such that $W_i(b) = \tilde{X}_i(b) = X_i$ and $I(\tilde{X}_i, \tilde{X}_i) \geq I(W_i, W_i)$. Hence

(2)

$$\begin{aligned}
D^2_{\rho M}(X_1, X_1) &= X_1(W_1\rho M) - (D_{W1}W_1)(\rho M) \\
&= X_1 < W_1, grad\rho M > - < D_{W_1}W_1, grad\rho M > \\
&= X_1 < W_1, \gamma_{1*}(t) > - X_1 < W_1, \gamma_{1*}(t) > + < W_1, D_{W_1}\gamma_{1*} > \\
&=< W_1, D_{\gamma 1*(t)}W_1 > (b) \\
&= \int_0^b \frac{d}{dt} < W_1, D_{\gamma 1*}W_1 > dt + < W_1, D_{\gamma 1*}W_1 > \\
&= \int_0^b |DW_1| - < W_1, D^2W_1 > dt + < W_1(0), S_{\gamma 1*(0)}W_1(0) > \\
&= \int_0^b |DW_2| - < W_1(t), R(W_1(t), \gamma_{1*}(t))\gamma_{1*}(t) > dt \\
&\quad + < W_1(0), S_{\gamma 1*(0)}W_1(0) > .
\end{aligned}$$

Similary,

(3)

$$D^2_{\rho N}(X_2,X_2)=\int_0^b |DW_2|^2 - < W_2(t), R(W_2(t),\gamma_{2*}(t)\gamma_{2*}(t) > dt$$
$$+ < W_2(0), S_{\gamma 2*(0)} W_2(0) > .$$

To compare the two integrals in (2) and (3), let a new vector field W^* be defined along γ_2 as follows. Let $\{e_1,i\}_{1\le i\le m}$ and $\{e_2,j\}_{1\le j\le n}$ be orthonormal because of M_p and N_q such that $e_{1,1}=\gamma_{1*}(0)$ *and* $e_{2,1}=\gamma_{2*}(0)$ respectively. Let $\{e_1,i(t)\}$ and $\{e_2,j(t)\}$ be their parallel translates along γ_1 and γ_2 respectively. Without loss if generality, we may assume that $e_{1,m}(b)=W_1(b)=(X_1)$ *and* $e_{2,m}(b)=W_2(b)=(X_2)$.

Define functions $\{h_i\}_{1\le i\le m}$ on $[0,b]$by $W_1(t)=\sum_{i=1}^m h_i(t)e_{1,i}(t)$. Then we define $W^*(t)=\sum_{i=1}^m h_i(t)e_{2,i}(t)$ for all $t\in[0,b]$ It follows that $W^*(b)=e_{2,m}(b)=W_2(b)=X_2$. Since there are no focal point of q along γ_2 and W_2 is a ∂N-jacobi field, by Theorem 1.1

$$D^2_{\rho N}(X_2,X_2)\le\int_0^b |DW^*|^2 - < R(W^*(t),\gamma_{2*}(t))\gamma_{2*}(t), W^*(t) > dt$$
$$+ < W^*(0), S_{\gamma 2*(0)} W^*(0) > .$$

Since $|DW^*|^2=\sum_{i=1}^m |h'(t)|^2=|DW_1|^2, |W^*|=|W_1|, W^*\perp\gamma_{2*}$ and $W_1\perp\gamma_{1*}$, the hypothesis implies that

$$< R(W^*(t),\gamma_{2*})\gamma_{2*}(t), W^*(t) > \ \ge\ < R(W_1(t),\gamma_{1*}(t))\gamma_{1*}(t), W_1(t) >$$

and

$$< W^*(0), S_{\gamma_{2*}} W^*(0) > \ \le\ < W_1(0), S_{\gamma 1*(0)} W_1(0) >$$

Hence we obtain that

$$D^2_{\rho N}(X_2,X_2)\le D^2_{\rho M}(X_1,X_1)$$

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