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動的 問題의 空間-時間 有限要素解析

Space-Time Finite Element Analysis of Transient Problem

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ABSTRACT

A space-time finite element method was presented for time dependent problem. The method which treat both the space and time unformly were proposed and numerically tested. The weighted residual process was used to formulate a finite element method in a space-time domain based upon continuous Galerkin method. This method leads to a conditional stabie high-order accurate solver.

국문요약

動的 問題에 對한 空間-時間 有限要素法을 提示하였다. 이 方法은 空間과 時間을 同一한 變數로 취급하였으며 空間-時間 領域에서의 有限要素 展開에 있어서는 連續的 갤러킨 方法에 根據하여 加重餘分法을 以用하였다. 이 方法은 조건부 安定을 주는 高次元的 正確性을 주는 解法인 것이다.

1. Introduction

Space and time variation could be treated uniformly by formulating space-time finite element methods[1)~3)]. The idea is to use the method of weighted residuals to treat space and time in a uniform manner and thus integrating both the spatial and temporal variations of the unknown quantities simultaneously. This method generates a complete space-time finite element decretization which eliminates the need for any additional ordinary differential equation solver to resolve the temporal behavior of the problem. A wide spectrum of problems, in cluding heat conduction, elastodynamics, and advective-diffusive sys-

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tems associated with fluid dynamics have been considered using space-time finite elememt methods in which the unknown quantites were assumed to be continuous with the respect to time[4)~7)].

2. Space-Time Domain

The space-time domain of such problems is the open spatial interval Ω decribed by

$$\Omega = \{x : 0<x<L\} \cdots\cdots (1)$$

and the open temporal interval I given by

$$I = \{t : 0<t<T\} \cdots\cdots (2)$$

Then the space-time domain is the product space $\Omega \times 1$.

This domain will be divided into time slabs as indicated in Fig. 1 Referring to Fig. 2, a space-time slab is then defined as

$$G_n = \Omega_n \times I_n \cdots\cdots (3)$$

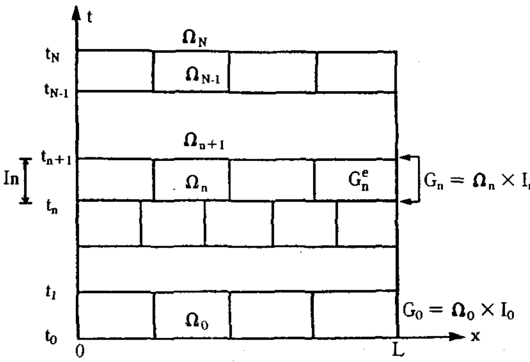

Fig. 1 The time domain

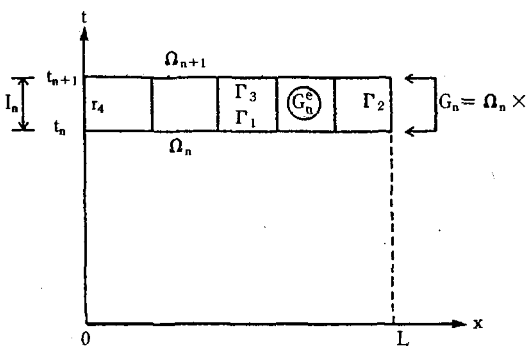

Fig. 2 A space-time slab

That is, a space-time slab is the first row corresponding to the interval $0 \leq t \leq t_1$, the second $t_1 \leq t \leq t_2$, and so on. Let n_{el} denote the number of space-time elements and the spatial domain be subdivided into n_{el} elements, Ω_n^e. e $= 1, 2, 3, \cdots\cdots n_{el}$. Then, for the n-th space-time element the domain is

$$G_n^e = \Omega_n^e \times In \quad e = 1, 2, 3, \cdots\cdots, n_{el} \cdots (4)$$

On G_n^e we take the temperature field $u(x, t)$ to be approximated by

$$u^e(x, t) = \Sigma\ N_k(G_n^e)uk \cdots\cdots (5)$$

where $N_k(G_n^e)$ are interpolating polynomials of degreer defined over the region G_n^e, uk are the nodal values of the field $u(x, t)$. The weighting function $W_k(G_n^e)$ are also defined over the space G_n^e.

3. Heat Conduction Boundary Value Problem

Heat conduction is governing by the linear partial differential equation given in Equation(6). This equation describes the unsteady temperature in an isotropic body occupying a domain $\Omega \times I$.

$$\rho c_p u(x, t) - \mathrm{div}(k \nabla u(x, t)) = f(x, t)$$

$$\text{in } \Omega \times I \cdots\cdots (6)$$

with boundary and initial conditons as follows :

$$u(0, t) = gb(t) \text{ on } \Gamma_4 \times I$$

$$k\nabla u(L, t) = -qb(t) \text{ on } \Gamma_4 \times I$$

$$k\nabla u(L, t) = -h_0(u(L)-u_\infty) \text{ on } \Gamma_2 \times I$$

$$u(x, 0) = u_0(x)$$

$$\text{in } \Omega(0),\ x \in \Omega \cdots\cdots (7)$$

Where u is the temperature field(℃) for $X \in \Omega$ at time $X \in I = [0, T]$ and where T is a given time, ∇ denotes the spatial gradient operator; k is the thermal conductivity of materials(W/m·℃); gb(t) is the specified function that give the temperature at the left end; qb(t) is the prescribed heat flux(W/m^2); ρ is the density of material(kg/m^3); c_p is the specific heat of material(W·sec/kg·℃); f is the heat generation rate(W/m^3); ho is the heat transfer coefficient

(W/m^2·℃) ; u_∞ is the temperature of the bulk fluid(℃), and the initial condition u_0 is a given function of x. Γ_2 and Γ_4 denote a noi-overlapping subdivision of the boundary Γ of Ω and $\acute{u} = \partial u/\partial t$. Equation(7) is also known as a Dirchlet condition and a Neumann condition.

4. Weighted Residual Formulation

On an element basis, an algebraic relation among the u_k can be obtained by the method of weighted residuals. The continuous Galerkin method for Equation(7) can now be places in a weighted residual formulation as seeking a function u^e :

$$\int_{Ge}\left[\frac{\partial}{\partial x}\left(k\frac{\partial u^e}{\partial x}\right)+f-\rho c_p\frac{\partial u^e}{\partial t}\right]w^e i\,(x,\ t)dG^e=0,\ i=1,\ 2,\cdots\cdots,\ NN \quad \cdots\ (8)$$

where G^e is the domain for element e and $w^e i$ are the weight functions and NN is the number of nodes in element G^e.

$$\int_{Ge}\left[\rho c_p\ u^e\frac{\partial w^e i}{\partial x}-k\frac{\partial u^e}{\partial x}\frac{\partial w^e i}{\partial x}\right]dG^e$$
$$+\int_{\Gamma e} nxk\frac{\partial u^e}{\partial x}w^e id\Gamma^e$$
$$-\int_{\Omega e}\rho c_p w^e i(t_n+1)u^e(t_n+1)d\Omega^e$$
$$+\int_{Ge} f\ w^e i\ dG^e$$
$$i=1,\ 2,\cdots\cdots,\ NN \quad \cdots\cdots\ (9)$$

5. Finite Element Formulation

5.1 Discretization of the Domain

The solution domain is discretized in the current problems using two-dimensional bilinear quadrilateral elements for a general region specified by a global co-ordinate system(Y, t) as shown in Fig. 3.

This transformation from the x-t region to the 2 × 2 square in natural co-ordinates system(ζ, η) can be expresses in matrix form as follow

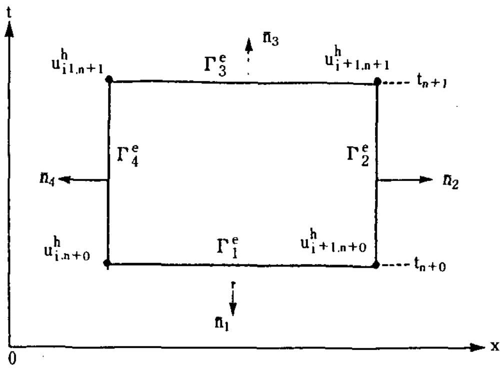

Fig. 3 Bilinear quadrilateral element in space-time region

$$\begin{Bmatrix} x \\ t \end{Bmatrix} = [N]\{c\} \quad \cdots\cdots\ (10)$$

where the interpolation function Ni(ζ, η) for node i has the following form

$$Ni(\zeta,\ \eta)=\frac{1}{4}(1+\zeta\zeta_i)(1+\eta\eta_i) \quad \cdots\ (11)$$

The unknown temperature is interpolated by the same functions, making this an isoparametric formulation.

$$u^e(x,\ t) = [N(\zeta,\ \eta)]\{u\} \quad \cdots\cdots\ (12)$$

Thus

$$u^e = \sum_{j=1}^{4} Nj\ uj \quad \cdots\cdots\ (13)$$

If the weight functions $w^h_i(x,\ t)$ are taken as the interpolation funcitons N_i, then Equation(9) becomes

$$\rho c_p\sum_{j=1}^{4}\int_{Ge} Nj\ uj\ Ni,\ t\ dG^e$$
$$-k\sum_{j=1}^{4}\int_{Ge} Nj,\ x\ uj\ Ni,\ x\ dG^e$$
$$-\int_{\Gamma e}\hat{n}x\ k\ Nj,\ x\ Ni\ uj\ d\Gamma$$
$$-\rho c_p\sum_{j=1}^{4}\int_{Ge} Nj(t_n+1)Nj(t_n+1)ujd\Omega^e$$
$$+\sum_{j=1}^{4}\int_{Ge} f\ Ni\ dG^e=0\ i=1,\ 2,\ 3,\ 4 \quad \cdots\cdots\ (14)$$

Equation(14) is rewritten in the form as

$$[C]\{u\} - [k]\{u\} + [B]\{u\} - [BT]\{u\} + \{q\} = 0 \quad \cdots\cdots (15)$$

or

$$[S]\{u\} + \{q\} = 0 \quad \cdots\cdots (16)$$

where

$$[S] = [C] - [K] + [B] - [BT] \quad \cdots\cdots (17)$$

Each individual matrix is given as

$$[C] = \rho c_p \int_{Ge} Nj\ Ni,\ t\ dG^e \quad \cdots\cdots (18)$$

$$[K] = k \int_{Ge} Nj,\ x\ Ni,\ x\ dG^e \quad \cdots\cdots (19)$$

$$[B] = k \int_{\Gamma e} \hat{n}x\ Nj,\ x\ Ni,\ d\Gamma^e \quad \cdots\cdots (20)$$

$$[BT] = \rho c_p \int_{Ge} Ni(t_n + 1)\ Nj(t_n + 1)\ d\Omega^e \quad \cdots\cdots (21)$$

$$\{q\} = \sum_{j=1}^{4} \int_{Ge} f\ NidGe \quad \cdots\cdots (22)$$

The integration will be performed numerically in the $\zeta - \eta$ space. The integral in the x-t space is transformed as follows :

$$[K] = \sum_{i=1}^{p} \sum_{j=1}^{p} \bar{W}i\bar{W}jk[N'(\zeta i,\ \eta j)]^t \{\Lambda 1\}^t \{\Lambda 1\} [N'(\zeta i,\ \eta j)] |J(\zeta i,\ \eta j)| \quad \cdots\cdots (23)$$

$$[K] = \sum_{i=1}^{p} \sum_{j=1}^{p} \bar{W}i\bar{W}j \rho c_p [N'(\zeta i,\ \eta j)]^t \{\Lambda 2\} [N'(\zeta i,\ \eta j)] |J(\zeta i,\ \eta j)| \quad \cdots\cdots (24)$$

$$[BT] = \rho c_p \sum_{i=1}^{p} \bar{W}i [N(\zeta i)] \{N(\zeta i)\} |J(\zeta i)| \quad \cdots\cdots (25)$$

$$[B_2] = k \sum_{i=1}^{p} \bar{W}i \{N(\eta j)\} \{\Lambda 1\} [N'(\eta j)] |J(\eta j)| d\eta \quad \cdots\cdots (26)$$

$$[B_4] = k \sum_{i=1}^{p} \bar{W}i \{N(\eta j)\} \{\Lambda 1\} [N'(\eta j)] |J(\eta j)| d\eta \quad \cdots\cdots (27)$$

Where $\bar{W}i$ and $\bar{W}j$ are Gauss weights, ζi and ηj are the co-ordinates of the Gauss points, and p is the number of Gauss points in each integration direction.

Substituting the functions and equations into the variation Equation(9) leads to the following element equation

$$[M]\{U\}_f = \{q\} \quad \cdots\cdots (28)$$

where

$$[M] = [C] - [K] - [BT] + [B]$$

$$\{u\}_f = [ui,n+0\ \ ui+1,n+0\ \ ui,n+1\ \ ui+1,n+1]^t \quad \cdots\cdots (29)$$

The matrices in this system show the temporal coupling between times n + 0 and n + 1 with initial time n + 0. Denoting by the subscript I and u quantites times n + 0 and n + 1, respectively, Equation(28) can be written in terms of partitioned matrices for each elements as follow :

$$[M_{00}]\{u\}_{n0+0} + [M_{n0}]\{u\}_{n+1} = \{q_1\}_{n+0} \quad \cdots\cdots (30)$$

and

$$[M_{0n}]\{u\}_{n0+0} + [M_{nn}]\{u\}_{n+1} = \{q_n\}_{n+1} \quad \cdots\cdots (31)$$

where $[M_{00}]$, $[M_{n0}]$, $[M_{0n}]$, and $[M_{nn}]$ are the property matrices related to time levels n + 0, and n + 1 at node i and i + 1, respectively.

5.2 Assembling the Finite Element Element Equation in Space and Time

At t = 0, we know all values of u as shown in Fig. 4, We need to assemble these individual element equations so that the values of u can be calculated at the end of the first time step.

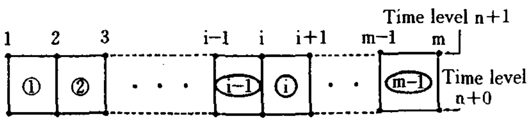

Fig. 4 Assemblage of elements on time slab

This continues so that during the (n + 0)-th step the values of u at ui, n + 0 are known and ui, n + 1 must be calculated.

The assembled equations of Equation(30) and (31) are expressed in matrix form as

$$[\bar{M}_{00}]\{u\}_{n0+0}+[\bar{M}_{n0}]\{u\}_{n+1} = \{Q_1\}_{n+0} \quad \cdots\cdots (32)$$

$$[\bar{M}_{0n}]\{u\}_{n0+0}+[\bar{M}_{nn}]\{u\}_{n+1} = \{Q_u\}_{n+1} \quad \cdots\cdots (33)$$

The matrices $[\bar{M}_{00}]$, $[\bar{M}_{n0}]$, $[\bar{M}_{0n}]$, and $[\bar{M}_{nn}]$ are tridiagonal.

We may written the Equation(32) and (33) as

$$[[\bar{M}_{n0}]-[\bar{M}_{00}][\bar{M}_{0n}]^{-1}[\bar{M}_{nn}]\{u\}_{n+1} = \{Q_u\}_{n+1}-[\bar{M}_{00}]\{Q_u\}_{n+1}[\bar{M}_{0n}]^{-1} \quad \cdots\cdots (34)$$

Equation(34) is rewritten in the final form as

$$[\bar{S}]\{u\}_{n+1}=\{Q\} \quad \cdots\cdots (35)$$

where

$$[\bar{S}]=[\bar{M}_{n0}]-[\bar{M}_{00}][\bar{M}_{0n}]^{-1}[\bar{M}_{nn}]$$

$$\{Q\}=\{Qu\}_{n+1}-[\bar{M}_{00}]\{Q_u\}_{n+1}[\bar{M}_{0n}]^{-1}$$

$[\bar{S}]$ is the assemblage property matrix; matrix related to time level; $\{u\}$ is the assemblage vector of nodal unknows, and $\{Q\}$ is assemblage vector of nodal force.

6. Numerical Implementations and Discussions

A finite steel bar of a uniform cross-section is initially at a uniform temrerature of 30℃. The bar is assumed to be thermally insulated on the circumference so that heat can only flow along the longitudinal dorection specified by the x coordinate. The bar is also assumed to have no internal heat generation. The thermophysical properties used for computation are as follow; thermal conductivity($k = 144.5$Cal/m·sec·℃); density($\rho = 7.875$g/cm^3); specific heat($c_p = 0.103$Cal/g·℃). Cases which involve three distinct boundary conditions were studied.

Case 1) Prescribed temperature, where

$u(0, t)=100$℃ $\quad u(L, t) = 20$℃ $\cdots$ (36)

Case 2) Prescribed heat flux, where

$q(0, t)=0 \quad u(L, t) = 20$℃ $\cdots$ (37)

Case 3) Convective heat transfer, where

$q(0, t)=0 \quad h_0 = 1440$Cal/m^2·sec·℃

$u_\infty=100$℃ $\cdots\cdots$ (38)

In order to investigate the effect of grid-spacing and time steps on the results, runs have been made for twenty elements which are uniformly discretized and for values t, ranging from 7 to 100sec. For simplicity. the bar is considered to be a unit length.

For case 1, the basic time increment was chosen to bo 7 sec., which corresponds to $\Delta t< \rho c_p (\Delta x)^2<2k$. Compared with the solutions obtained by using the analytical method, the accuracy of the present results is favorable, For Cases 2 and 3, the large time increment $\Delta t = 28$sec. is taken for numerical stability considerations. It is noted that in all cases the solution is very accurately predicted, even with 20-element model and the large time step.

However, for and accurate prediction of the temperature, a finer finite element discretization and smaller time step Δt need to be employed. For each case, it was found that the differences reduce rapidly at times beyond first time step. As shown in the figures, no significant instability problems and much more repid convergence to the analytical solution were experienced in this approach.

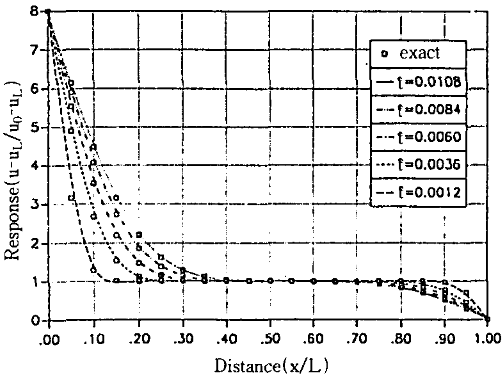

Fig. 5 Case 1 : Comparison of exact and continuous Galerkin solutions for a finite bar with $\Delta t = 7$sec

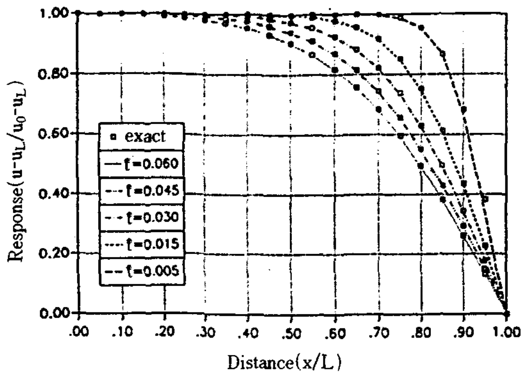

Fig. 6 Case 2 : Comparison of exact and continuous Galerkin solutions for a finite bar with $\Delta t = 28$sec

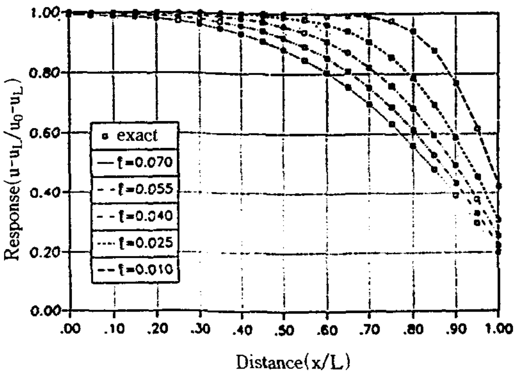

Fig. 7 Case 3 : Comparison of exact and continuous Galerkin solutions for a finite bar with $\Delta t = 28$sec

7. Conclusion

The aim of this research was to investigate the development of a space-time finite element formulation capable of solving a wide range of time-dependent problems. We have demonstrated the efficiency of using finite elements in both time and space where we consider the time-space domain as a whole in the generation of finite elements. The most frequently used method for the numerical integration of parabolic differential equations is the method of lines, where one first uses a discretization of space derivatives by finite differences and finite elements and then uses a time-stepping method for the solution of the resulting system of ordinary differential equations. Such methods are, at least conceptually, easy to perform. One disadvantage of this approach is that it separates the spatial and temporal coupling inherent in time-dependent problems. Moreover, they can be expensive if steep gradients occur in the solution; stability must be controlled, and the global error control can be troublesome.

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